ALGEBRA

By the same Authors

COORDINATE GEOMETRY

by C. O. Tuckey and F. J. Swan

GEOMETRY FOR SIXTH FORMS

ALGEBRA

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PREFACE

THE aim of this book is to provide a course of Algebra suitable for those who continue to study mathematics beyond O level of the General Certificate of Education.

The contents should satisfy the needs of all intending to take A level in the G.C.E. with the exception of those who are preparing for Mathematical Scholarships to the University, who will need to supplement the later chapters, and of those who wish to offer Statistics in the G.C.E. for whom the authors feel that a separate text-book dealing exclusively with Statistics is needed.

The first few chapters provide revision of elementary parts of the subject and include methods and items not usually met in preparation for O level; two-row determinants are introduced in solving simultaneous equations, the quadratic function is given very full treatment and graphs of functions involving the modulus sign are discussed.

The reports, prepared for the Mathematical Association on Algebra and Analysis (parts of both of which subjects are included in School Algebra) have been consulted and many valuable suggestions adopted, especially in the treatment of Partial Fractions and in dealing with the Standard Series.

It is hoped that the chapter on Complex Numbers will be found sufficiently complete for those Science students who need an introduction to the subject; also that the final chapters on Inequalities, Determinants and Theory of Numbers will not only cover the demands of those Examining Boards which include these subjects in their syllabuses, but will also be found a useful reservoir of material for those carrying on with the subject after taking their A level examinations.

Almost every section of the text is followed by a batch of examples, and a large number of miscellaneous examples and test papers are included.

The authors have to thank the authorities of Bristol University, London University, the Northern Joint Board and the Oxford and Cambridge Joint Board for permission to include questions (marked B, L, N, O. & C. respectively) from papers set by them.

C. O. T.
W. A.

CONTENTS

CHAPTER I

REVISION AND SOME EXTENSIONS

THE revision of a number of familiar processes in algebra, with somewhat harder examples, and extension of these processes to cases often excluded from the elementary course are the purposes of this chapter and the next.

Chapters III and IV are also partly revision.

Some specimen examples are worked first.

Simple equations

Example I. Solve the equation :

$$\frac{3x+2}{5}-\frac{2x-1}{3}=\frac{x}{18}.$$

The L.C.M. of 5, 3, 18 is 90 ; multiply both sides of the equation by 90.

$$\therefore\ \frac{90(3x+2)}{5}-\frac{90(2x-1)}{3}=\frac{90\cdot x}{18};$$

$$\therefore\ 18(3x+2)-30(2x-1)=5x;$$

$$\therefore\ 54x+36-60x+30=5x;$$

$$\therefore\ -11x=-66;$$

$$\therefore\ x=6.$$

Check : If $x=6$ L.H.S. $=\frac{20}{5}-\frac{11}{3}=4-3\frac{2}{3}=\frac{1}{3}$;

R.H.S. $=\frac{6}{18}=\frac{1}{3}$.

Example II. Solve for x the equation $a(x-a)=b(x-b)$ where a, b are unequal constants.

$$\therefore\ ax-a^2=bx-b^2;$$

$$\therefore\ ax-bx=a^2-b^2;$$

$$\therefore\ (a-b)x=(a-b)(a+b);$$

$$\therefore\ x=a+b.$$

Check : L.H.S. $=a(b)=ab$; R.H.S. $=b(a)=ab$.

Examples 1. Solve the equations :

1. $\frac{2(x+1)}{5}-8=\frac{2x}{16}-1$.

2. $\frac{1}{2}(x-2)-\frac{1}{4}(x+2)=\frac{1}{3}(x-3)$.

3. $\frac{x+1}{2}-\frac{1}{4}=x-\frac{2x-1}{3}$.

4. $\frac{x}{7}+\frac{x-1}{2}-x+1=0$.

5. $\dfrac{x-7}{8}=\dfrac{x-3}{11}$.

6. $\dfrac{x+5}{12}=\dfrac{1-6x}{5}$.

7. (i) $3(x-1\cdot8)+2(x+2\cdot4)=5\cdot9$; (ii) $5(7\cdot3-x)=26+9(x+\cdot7)$.

8. (i) $\dfrac{x-5}{2x-11}=\dfrac{2}{3}$; (ii) $\dfrac{1}{x+1}=\dfrac{4}{2x-6}$.

9. Solve for x, (i) $2a(x-a)-b(x-2b)=3ab$; (ii) $\dfrac{x}{a}+\dfrac{x}{b}=1$.

10. If $x=\dfrac{a-b}{c-d}$, prove that $cx+b=dx+a$.

11. Solve $ax-b=d-cx$.

12. The two equations $\begin{cases} px+q=rx+p \\ qx+p=px+r \end{cases}$ give the same value of x; show that unless $q=r$ it is necessary that $p=\frac{1}{2}(q+r)$.

Simultaneous simple equations

To find the values of two letters x and y there must be two equations satisfied simultaneously.

Example I. Solve

$$5x-2y=7, \quad \text{.......(i)}$$
$$3x+5y=29. \quad \text{.......(ii)}$$

1st Method.

Multiply both sides of (i) by 5: $25x-10y=35$.
Multiply both sides of (ii) by 2: $6x+10y=58$.

Add: $31x=93$;

$\therefore\ x=3$.

Substitute in (i): $15-2y=7$; $\therefore\ y=4$.

Check in (ii): L.H.S. $=9+20=29=$ R.H.S.;

$\therefore\ x=3,\ y=4$.

2nd Method.

From (i) $x=\dfrac{7+2y}{5}$, an from (ii) $x=\dfrac{29-5y}{3}$;

$$\therefore\ \frac{7+2y}{5}=\frac{29-5y}{3}.$$

Multiply by 15: $21+6y=145-25y$;

$\therefore\ 31y=124$;

$\therefore\ y=4$;

$$\therefore\ x=\frac{7+8}{5},\ \text{i.e. } x=3.$$

Example II. Solve

$$\frac{x}{3}-\frac{y}{2}=1, \quad \text{.......(i)}$$
$$\frac{y}{11}=11-\frac{x}{4}. \quad \text{.......(ii)}$$

Clearing each equation of fractions (multiplication by 6 and by 44),

$$2x-3y=6,$$
$$4y=484-11x.$$

To get $12y$ in each equation, multiply by 4 and by 3:

$$8x-12y=24,$$
$$12y=1452-33x.$$
$$\text{Add}: \quad 8x=1476-33x;$$
$$\therefore\ 41x=1476;$$
$$\therefore\ x=36.$$

Substituting in (i): $12-\dfrac{y}{2}=1$; $\therefore\ y=22.$

Check in (ii): $\dfrac{y}{11}=2$; $11-\dfrac{x}{4}=11-9=2.$

$$\therefore\ x=36,\ y=22.$$

Examples 2. Solve the simultaneous equations:

1. $6x+8y=35$, $2y-4x=6$.

2. $2x+5y=-3$, $7x-10y=17$.

3. $2x+y=-12$, $x-7y=9$.

4. $8x+12y=20$, $18y-5x=-16$.

5. $4(x-2)=3(y+2)$, $3(x+2)=7(y+1)$.

6. $5x-6y=17$, $y=x-10$.

7. $x-y=2x-3y=6$.

8. $2x+3y=2=10x-9y$.

9. $\frac{1}{2}(x+y)=4\frac{1}{2}$, $\dfrac{x}{4}-y=1$.

10. $x+y=3$, $\dfrac{x}{7}+y=0$.

11. $\cdot3x+\cdot5y=\cdot23$, $6x+5y=2\cdot6$.

12. $3x+2y=9\cdot29$, $x-2y=\cdot43$.

13. $\dfrac{x}{3}-\dfrac{2x-y}{4}=-2$, $\dfrac{x-3y}{6}=5-2(5y-x)$.

14. $\dfrac{x-y}{2}=\dfrac{2x}{5}=6-y$.

15. $\cdot3(7x+2y)-\cdot08=\cdot7(x+5)$, $y-\cdot2x=3(y-1)$.

16. $cx-by=ac$, $x+y=a+b+c$.

17. $ax+by=ac$, $bx-ay=bc-a^2-b^2$.

Examples 3. Practice in simple problems:

1. How old is a man if his age 12 years hence will be double what his age was 15 years ago?
2. When a son will be as old as his father is now, the father will be four times as old as the son is now. What is the ratio of their present ages?
3. Four subtracted from a quarter of a number gives the same result as subtracting the number from two and then dividing by three; find the number.

4. A boy is a minute late if he walks to school at the rate of 11 yards in 9 seconds. If he had walked at the rate of 22 yards in 15 seconds he would have been $\frac{1}{2}$ minute too early. How far had he to walk?
5. A pint of water weighs 1·25 lb. and a pint of milk weighs 1·29 lb. A sample from a 14-gallon can full of supposed milk was found to weigh 1·285 lb. per pint. How many pints of water were in the 14-gallon can?
6. Interest on £650 at $x\%$ together with interest on £400 at $y\%$ totals £40. If x and y are interchanged, the total is only £38 15s. Find x and y.
7. Two railway passengers have to pay 2s. 11d. and 2s. 1d. excess on their respective luggage. The total weight of the luggage was 320 lb., and if it had belonged to one passenger the excess charge would have been 9s. 2d. What weight of luggage is a passenger allowed free of charge?
8. A man walked from A to B at 5 m.p.h. and rode from B to C at 20 m.p.h., taking altogether 1 hr. 39 min. for the journey from A to C. If he had ridden from A to B and walked from B to C, the rate being the same for walking and riding as before, he would have taken 2 hr. 6 min. for the whole journey. Find the distance from A to C.
9. It is now 9 a.m., and if I cycle to the station at 12 m.p.h. I shall be $22\frac{1}{2}$ minutes early for my train, while if I walk at 4 m.p.h. I shall be $2\frac{1}{2}$ minutes late. How far is it to the station and what time is the train?

The General Case for Two Linear Equations

Solve
$$a_1x + b_1y + c_1 = 0 \quad\ldots\ldots(1)$$
$$a_2x + b_2y + c_2 = 0 \quad\ldots\ldots(2)$$

[a_1, b_1, c_1, a_2, b_2, c_2 are constants; the use of suffices in this way extends the alphabet to any desired extent.]

Solution: $(1) \times b_2$: $a_1b_2x + b_1b_2y + c_1b_2 = 0.$

$(2) \times b_1$: $a_2b_1x + b_1b_2y + c_2b_1 = 0.$

Subtract : $(a_1b_2 - a_2b_1)x + c_1b_2 - c_2b_1 = 0.$

$$\therefore\ x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{(see } N.B. \text{ below)}$$

$(2) \times a_1$: $a_1a_2x + a_1b_2y + a_1c_2 = 0.$

$(1) \times a_2$: $a_1a_2x + a_2b_1y + a_2c_1 = 0.$

Subtract : $(a_1b_2 - a_2b_1)y + a_1c_2 - a_2c_1 = 0.$

$$\therefore\ y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}.$$

The student should notice the cyclic order b, c; c, a; a, b which perhaps has been noticed in formulae in Trigonometry.

N.B. This assumes that $a_1b_2 - a_2b_1$ is not zero.

If $a_1b_2 - a_2b_1 = 0$, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$; suppose each of these fractions $= k$, and so $a_1 = ka_2$, $b_1 = kb_2$.

Hence the equations are $ka_2x + kb_2y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$.

Consequently if $c_1 = kc_2$ the equations are the same

while if $c_1 \neq kc_2$ the equations are inconsistent,

and so in neither case is it possible to determine x and y.

It is useful to learn to write down at once the solution of (1) and (2) by using the following scheme for the coefficients:

$$\begin{array}{ccccccccc} a_1 & & b_1 & & c_1 & & a_1 & & b_1 \\ & \times & & \times & & \times & & \times & \\ a_2 & & b_2 & & c_2 & & a_2 & & b_2 \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Multiply as indicated by the arrows, and write the result as

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

cover up the first column of (A) and take $\begin{array}{ccc} b_1 & & c_1 \\ & \times & \\ b_2 & & c_2 \end{array}$ to mean $(b_1c_2 - b_2c_1)$; then covering up the first two columns gives $\begin{array}{ccc} c_1 & & a_1 \\ & \times & \\ c_2 & & a_2 \end{array}$ which is read as $(c_1a_2 - c_2a_1)$, and covering the first three columns gives $\begin{array}{ccc} a_1 & & b_1 \\ & \times & \\ a_2 & & b_2 \end{array}$ i.e. $(a_1b_2 - a_2b_1)$.

This use of the array of coefficients (A) suggests the notation $\begin{vmatrix} b_1 & c_2 \\ b_2 & c_2 \end{vmatrix}$ for $(b_1c_2 - b_2c_1)$, and introduces the *determinant* notation which is of great importance in Algebra.

We define the 2-row or 2×2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ to mean $(ad - bc)$ and so the result (B) may be written as

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \quad \ldots\ldots\ldots\ldots\ldots\text{(C)}$$

Note. To solve equations in this way, *they must be written with zero on the R.H.S.*

Some teachers prefer, and some students find it easier, to write this result as

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad \ldots\ldots\ldots\ldots\text{(D)}$$

obtaining the 2×2 determinants in the denominators from the array of coefficients

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}$$

and merely covering up the columns in turn.

If (D) is adopted, then care must be exercised in inserting the *minus* sign; it would be as well to choose one of the methods and stick to it so as to avoid confusion.

Applying the solution (C) to the equations in Example I, we rewrite the equations as

$$5x - 2y - 7 = 0,$$
$$3x + 5y - 29 = 0,$$

and get

$$\frac{x}{\begin{vmatrix} -2 & -7 \\ 5 & -29 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -7 & 5 \\ -29 & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 5 & -2 \\ 3 & 5 \end{vmatrix}},$$

i.e.

$$\frac{x}{58+35} = \frac{y}{-21+145} = \frac{1}{25+6};$$

$$\therefore\ x = \frac{93}{31} = 3, \quad \text{and} \quad y = \frac{124}{31} = 4.$$

The extension from the two-row or 2×2 determinant to the three-row or 3×3 determinant is given in Chapter XIV; some may prefer to take parts of that chapter at an early stage.

Examples 4

1. Work out the determinants:

(i) $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$; (ii) $\begin{vmatrix} 2 & 10 \\ 2 & 7 \end{vmatrix}$; (iii) $\begin{vmatrix} -1 & -4 \\ 2 & 6 \end{vmatrix}$; (iv) $\begin{vmatrix} 3 & -7 \\ 4 & -8 \end{vmatrix}$.

Solve the equations:

2. (i) $2x + 3y - 37 = 0$, $x - 13y - 4 = 0$; (ii) $2x + 5y = -3$, $7x - 10y = 17$;

(iii) $6x + 2y - 15 = 0$, $-4x + 11y + 9 = 0$; (iv) $12x + 7y = 59$, $8x + 13y = 81$.

3. Work out (i) $\begin{vmatrix} x & y \\ 2y & 3x \end{vmatrix}$; (ii) $\begin{vmatrix} 3a & 2a \\ 3x & 2x \end{vmatrix}$; (iii) $\begin{vmatrix} x & -y \\ y & x \end{vmatrix}$,

and prove that $\begin{vmatrix} (a-b) & (a+b) \\ (a+6b) & (a-7b) \end{vmatrix} = -15ab + b^2$.

4. Find the ratio $x : y : z$ if

$$a_1x + b_1y + c_1z = 0 \quad \text{and} \quad a_2x + b_2y + c_2z = 0.$$

5. Find the ratios $x^2 : x : 1$ if

$$a_1x^2 + b_1x + c_1 = 0 \quad \text{and} \quad a_2x^2 + b_2x + c_2 = 0.$$

[Since $x^2 : x$ must equal $x : 1$, a relation between the constants can be obtained.]

In Examples 6, 7, 8 find x and y by solving any two of the equations and show that these values satisfy the third.

6. $2x - 3y + 7 = 0.$
$3x + 5y - 6 = 0.$
$7x - y + 8 = 0.$

7. $5x + 2y - 11 = 0,$
$3x - 5y + 4 = 0,$
$19x - 11y - 10 = 0.$

8. $6x - 7y + 13 = 0,$
$5x + 8y - 2 = 0,$
$8x - 37y + 43 = 0.$

Elimination

In solving a pair of equations in two unknown quantities, it is usual to begin by using some device which will result in an equation involving one of the quantities and not the other one. The device used *eliminates* one of the quantities. For example, the first method on p. 2 begins by eliminating y while the second method eliminates x.

When told to eliminate one or more letters from some equations it is necessary to produce a relation which does not involve those letters.

One letter can be eliminated from two equations, two letters from three equations, and so on.

Example I. Eliminate t between the equations $x = at^2$, $y = 2at$.
Here $y^2 = 4a^2t^2$, i.e. $y^2 = 4a(at^2)$;
$\therefore\ y^2 = 4ax$, which is the *eliminant*.

Example II. Eliminate x and y between the equations

$$\begin{aligned} ax + by + c &= 0, \\ x + y + 1 &= 0, \\ a^2x + b^2y + c^2 &= 0. \end{aligned}$$

From the first pair of equations $x = \dfrac{b-c}{a-b}$, $y = \dfrac{c-a}{a-b}$, and so substituting in the third equation gives

$$a^2(b-c) + b^2(c-a) + c^2(a-b) = 0,$$

which is the eliminant.

[It has been assumed that $a \neq b$, for if $a = b$ it would be necessary for c to equal a if the first two equations were true; this would make the first two equations the same and the elimination impossible.]

Example III. Eliminate x between the equations

$$ax^2+bx+c=0,$$
$$px^2+qx+r=0.$$

Solution. As suggested in Examples 4, No. 5, we solve for $x^2 : x : 1$ as though x^2 and x were independent quantities. Using the determinant notation,

$$\frac{x^2}{\begin{vmatrix} b & c \\ q & r \end{vmatrix}}=\frac{x}{\begin{vmatrix} c & a \\ r & p \end{vmatrix}}=\frac{1}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}} \quad \text{or} \quad \frac{x^2}{br-cq}=\frac{x}{cp-ar}=\frac{1}{aq-bp}.$$

But $$\frac{x^2}{x}=\frac{x}{1}; \quad \therefore \frac{br-cq}{cp-ar}=\frac{cp-ar}{aq-bp}.$$

Clearing fractions $(br-cq)(aq-bp)=(cp-ar)^2$.
This is the required eliminant.

Example IV. Eliminate x from

$$a\cos x+b\sin x+c=0,$$
$$p\cos x+q\sin x+r=0.$$

Solution. Solving for $\cos x$ and $\sin x$ as though they were independent:

$$\frac{\cos x}{\begin{vmatrix} b & c \\ q & r \end{vmatrix}}=\frac{\sin x}{\begin{vmatrix} c & a \\ r & p \end{vmatrix}}=\frac{1}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}}$$

or $$\frac{\cos x}{br-cq}=\frac{\sin x}{cp-ar}=\frac{1}{aq-bp}.$$

But $\cos^2 x+\sin^2 x=1$;

$$\therefore (br-cq)^2+(cp-ar)^2=(aq-bp)^2.$$

Examples 5

1. Eliminate t between (i) $x=at$, $yt=b$; (ii) $x=\dfrac{a(t^2-1)}{t^2+1}$, $y=\dfrac{2bt}{t^2+1}$.
2. Find a from (i) $x^2+y^2=axy$; $x+y=5$, $2x-3y=0$;
   (ii) $x=3y$, $x^2-4y^2=80$, $x^2+3y^2=a^2-4$.
3. From the equations $v=u+at$, $s=ut+\frac{1}{2}at^2$ eliminate (i) u; (ii) t.
4. From the equations $t=s/c$, $s^2+c^2=1$ prove that $1/c^2-t^2=1$.
5. Eliminate x from the equations $2x+3y=11$, $x^2+y^2=30$.
6. Eliminate y from the equations $4x-3y=7$; $x^2-xy+y^2=13$.
7. Eliminate θ between $a\cos\theta=p$, $a\sin\theta=q$, $\cos^2\theta+\sin^2\theta=1$.
8. Eliminate x from $ax^2+2bx+3c=0$, $bx^2+2ax+4d=0$.
9. Eliminate x and y between

$$ax+hy+g=0, \quad hx+by+f=0, \quad gx+fy+c=0.$$

10. If $a\cos x+b\sin x+c=0$ and also $5\cos x+3\sin x+2=0$, show that $5a^2+21b^2-34c^2+12bc+20ca-30ab=0$.

11. Use $\sec^2 x - \tan^2 x = 1$ to eliminate x from

$$a \sec x - b \tan x = 1,$$
$$8 \sec x - 12 \tan x = 1.$$

12. If $\cos x + \sin x = a$ and $\cos^2 x - \sin^2 x = b$, show that $(a^2-1)^2 + b^2 = 1$.

13. If $a \sec x + b \tan x = f$ and $b \sec x + a \tan x = g$, show that

$$a^2 - b^2 = f^2 - g^2.$$

Simultaneous Equations in Three Unknowns

Three equations of the first degree in three unknowns are solved by a method essentially the same as that used for two equations in two unknowns.

Example I. Solve
$$\begin{aligned} x + y + z &= 13 \ \ldots \ \text{(i)},\\ 3x + y - 3z &= 5 \ \ldots \ \text{(ii)},\\ x - 2y + 4z &= 10 \ \ldots \ \text{(iii)}, \end{aligned}$$

[We eliminate z *twice*.]

Multiplying (i) by 3 and adding to (ii),

$$6x + 4y = 44.$$

Multiplying (i) by 4 and subtracting (iii),

$$3x + 6y = 42.$$

From these two equations (doubling the last and subtracting the other),

$$8y = 40; \quad \therefore\ y = 5.$$

Substituting gives $3x + 30 = 42$; $\therefore\ x = 4$.
Substituting both these values in (i) gives $z = 4$.
Checking in (ii), $12 + 5 - 12 = 5$;
in (iii), $4 - 10 + 16 = 10$.
The answer is $x = 4$, $y = 5$, $z = 4$.

[*Note.* It will do equally well to eliminate x, or y, instead of z.
If x is chosen, multiply (i) by 3 and subtracting (ii), $2y + 6z = 34$,
(iii) by 3 and subtracting (ii) $-7y + 15z = 25$,
and these equations give $y = 5$, $z = 4$.]

Example II. Solve
$$\begin{aligned} 4x + y + z &= 2 \ \ldots \ \text{(i)},\\ x + 2y &= 1 \ \ldots \ \text{(ii)},\\ z - x - 5y &= 5 \ \ldots \ \text{(iii)}. \end{aligned}$$

Since (ii) does not contain z, eliminating z from (i) and (iii) gives the two equations in two unknowns which are needed.

Subtract (iii) from (i): $5x + 6y = -3$
This with $x + 2y = 1$ gives $x = -3$, $y = 2$
Substitution in (i) gives $-12 + 2 + z = 2$; $\therefore\ z = 12$.
Check in (iii) $12 + 3 - 10 = 5$.
[The method is similar if there are more unknowns.]

Example III. Solve
$$\begin{aligned} x+y+z+w &= 12 \quad \ldots \text{(i)},\\ x-y+z &= 9 \quad \ldots \text{(ii)},\\ 2x+y &= 8 \quad \ldots \text{(iii)},\\ y-4z &= 5 \quad \ldots \text{(iv)}. \end{aligned}$$

Multiply (ii) by 4 and add (iv), $4x-3y=41$.
Take this with (iii); multiply (iii) by 3 and add to get $10x=65$.
$\therefore\ x=6\frac{1}{2}$, whence $y=-5$, $z=-2\frac{1}{2}$, $w=13$.

Examples 6. Solve the simultaneous equations:

1. $\left.\begin{aligned} 2x-y+z&=3\\ x+2y+z&=12\\ 4x-3y+z&=1 \end{aligned}\right\}$

2. $\left.\begin{aligned} x-3y+z&=10\\ 2x-7y-5z&=-2\\ x+y-2z&=5 \end{aligned}\right\}$

3. $\left.\begin{aligned} x+y&=2\\ y+z&=-2\\ z+x&=12 \end{aligned}\right\}$

4. $\left.\begin{aligned} x-y&=3\\ y-z&=5\\ z+x&=9 \end{aligned}\right\}$

5. $\left.\begin{aligned} x+y+z&=6\\ 2y+z&=9\\ z-2y&=1 \end{aligned}\right\}$

6. $\left.\begin{aligned} x+y+z+w&=12\\ x-y+z&=9\\ 2x+y&=8\\ y-4z&=5 \end{aligned}\right\}$

7. $\left.\begin{aligned} 3+x&=5-4w\\ z+x&=3w\\ 7w&=z+2 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3(z-1)&=2(y-1)\\ 4(y+x)&=9z-4\\ 2y&=7(5x-3z)+9 \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{x+1}{2}&=\frac{1}{3}+\frac{2y}{9}\\ x+\frac{y}{2}+4z&=\frac{1}{4}(y+2z+1)\\ x+2z&=1 \end{aligned}\right\}$

10. $\left.\begin{aligned} 7x-11y+2z&=10\\ -10x+3y+5z&=-15\\ 12x-y-6z&=31 \end{aligned}\right\}$

11. Show that the equations $\left.\begin{aligned} x-2y-4z&=3\\ 2x-y+5z&=4\\ 4x-5y-3z&=5 \end{aligned}\right\}$ have no solution.

12. Show that the equations $\left.\begin{aligned} x-2y-4z&=3\\ 2x-y+5z&=4\\ 4x-5y-3z&=10 \end{aligned}\right\}$ are satisfied by $x=\frac{1}{3}(5-14z)$, $y=-\frac{1}{3}(2+13z)$ for all values of z; explain how these equations are obtained.

13. Show that the equations $\left.\begin{aligned} x-y+z&=1\\ 3x+y-2z&=0 \end{aligned}\right\}$ are satisfied by $x=\frac{1}{4}(z+1)$, $y=\frac{1}{4}(5z-3)$ for all values of z.

Linear Equations and Graphs

If some work has been done on graphs it will be known how to mark points on graph paper when pairs of numbers, usually called

the x and the y of the points, are given; also that a first degree (or simple) equation connecting x and y will give a straight line graph, which is the reason for calling first degree equations *linear* equations.

When one linear equation in x and y is given, any value of x may be chosen and the corresponding value of y found from the equation.

There is no limit to the number of pairs of values which satisfy the equation, just as there is no limit to the number of points on a straight line.

But if two linear equations are given, there is only *one* pair of values of x and y which satisfy both of them, namely the coordinates of the point at which the two lines meet.

Example I. Draw the graph of $4y-3x=12$ and show that the points $(10, 10\frac{1}{2})$ and $(-6, -1\frac{1}{2})$ lie on it, both from the graph and from the equation.

What is the gradient of the line?

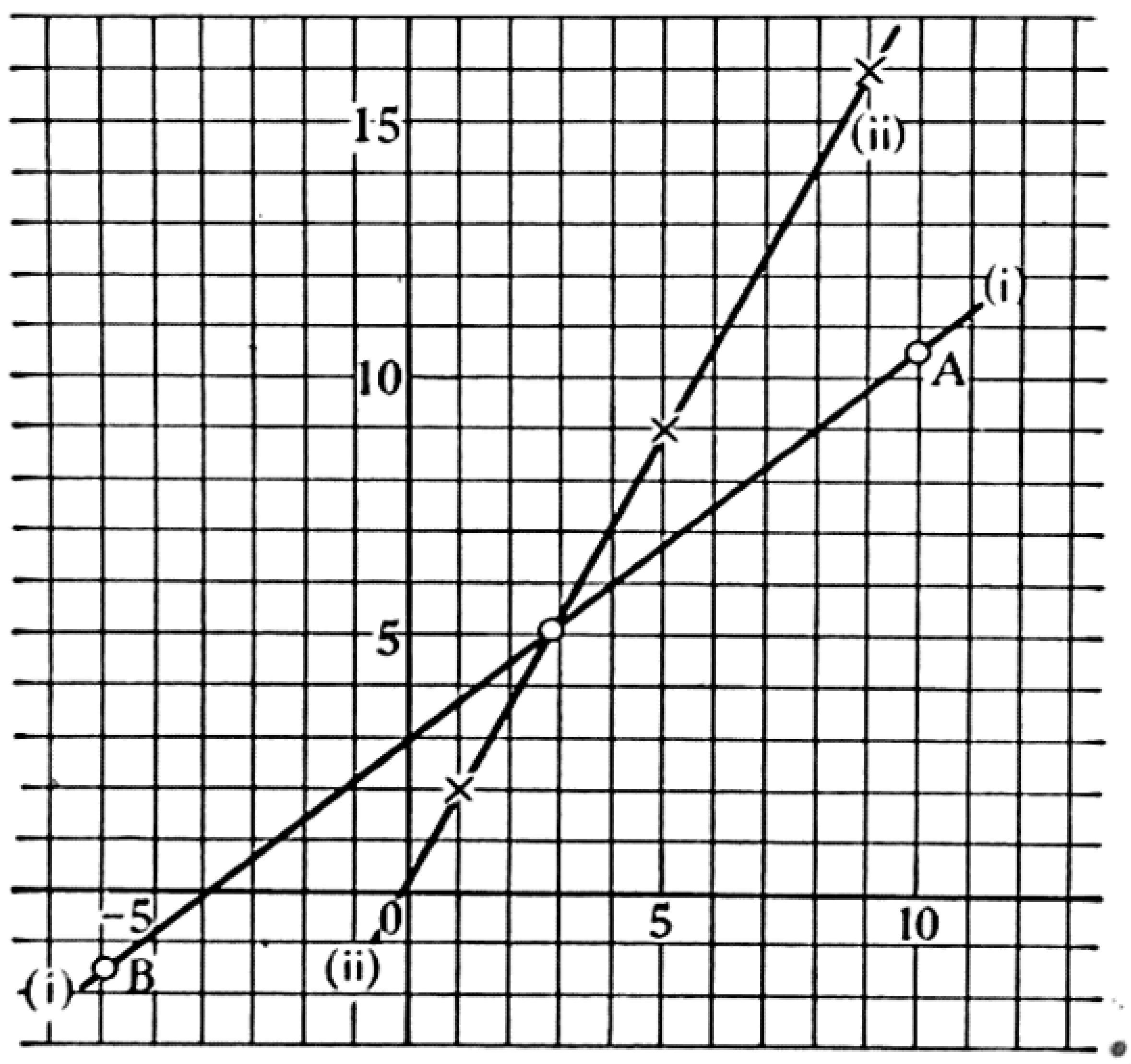

FIG. 1.

Writing the equation as $y=\frac{3}{4}x+3$ and taking values of x so as to avoid fractions we get the table

x	0	4	8	12
y	3	6	9	12

giving line (i) in the diagram.

Since y increases by 3 whenever x increases by 4, the gradient, defined as y-step/x-step is $\frac{3}{4}$.

The points $(10, 10\frac{1}{2})$, $(-6, -1\frac{1}{2})$ are shown on the line (i) at A and B.

For $(10, 10\frac{1}{2})$, $\quad 4y-3x = 42-30 = 12.$

For $(-6, -1\frac{1}{2})$, $\quad 4y-3x = -6+18 = 12.$

So both points lie on the line.

Examples 7

1. Solve the following pairs of equations. In each case verify the solution by drawing the graphs.

(i) $4y-3x=12$, $4y-7x=1$, $\quad$ (ii) $2x+y=5$, $x=3y+9$, $\quad$ (iii) $x-2y+4=0$, $x+y-8=0$.

[The graphs for (i) are shown in Fig. 1.

Graphical solution of linear equations is of little use. It is much simpler to check the solutions otherwise.]

2. Draw the graph of $x+y=5$ and show that $(0, 0)$ and $(4, 2)$ are on opposite sides of the line.

3. Draw the graph of $2y+3x=4$ and see which of the points $(6, -7)$, $(1, 1)$, $(-2, 5)$ are on the line.

Prove that the point $(3\frac{1}{3}, -3)$ should be on the line, and mark the point on your graph.

4. Draw with the same origin and axes the graphs of (i) $y=\frac{22}{7}.x$; (ii) $y=2{\cdot}54x$. How could these graphs be used as ready reckoners?

5. A set of marks of which the lowest is 63 and the highest 126 is to be adjusted so that the highest mark becomes 100 and the lowest 0. Show how a graph can be used to do this. What does a mark of 87 become?

Finding Equations for lines

Example I

Show that the table

x	1	5	9
y	2	9	16

gives points on a graph which lie on straight line; find the gradient of the line and its equation. [Shown as line (ii), Fig. 1.]

The quantity by which x increases (the x-step) between one entry and the next is 4; in each case the corresponding increase in y (the y-step) is 7. This shows that the points lie on a line whose gradient (defined as y-step/x-step) is 7/4. The equation of the line will be $y=\frac{7}{4}x+c$ where c is a number found by using one of the given points. Putting $x=1$, $y=2$, we get $2=\frac{7}{4}+c$, so $c=\frac{1}{4}$.

$\therefore$ the equation is $y=\frac{7}{4}x+\frac{1}{4}$ or $4y-7x=1$.

As a check, note that $4.9-7.5=1$ and $4.16-7.9=1$.

Another method is to suppose the equation of the line to be $ax+by=1$; substituting two pairs of values for x and y we get two equations for a and b. Using (1, 2) and (5, 9) we get $a+2b=1$, $5a+9b=1$, and these give $a=-7$, $b=4$. Substituting $x=9$, $y=16$ in $-7x+4y=1$ we find $-63+64=1$, as it should be.

Example II. Find the equation of the line given by $\begin{array}{c|c|c|} x & a & b \\ y & c & d \end{array}$.

The gradient, y-step/x-step, is $\dfrac{d-c}{b-a}$.

The equation is $y(b-a)=x(d-c)+e$ where to find e we put $x=a$, $y=c$.

$$\therefore\ c(b-a)=a(d-c)+e;$$
$$\therefore\ e=bc-ad;$$
$$\therefore\ \text{the equation is } y(b-a)=x(d-c)+bc-ad.$$

Check for (b, d), $\text{L.H.S.}=d(b-a)=db-da,$

$\text{R.H.S.}=b(d-c)+bc-ad=bd-\cancel{bc}+\cancel{bc}-ad=\text{L.H.S.}$

General expressions may be found for the coordinates of points which lie on a line whose equation is given as in the following example.

Example III. Find three pairs of whole numbers for x and y satisfying the equation $3x-2y=13$; also find general expressions for such values.

Solution. If x is a whole number it must be odd, since 13 is odd; trial gives $x=5$, $y=1$ as a solution. Also since $3x-2y$ does not change, y must change by 3 if x changes by 2.

$\therefore$ two other pairs of values are $(5+2, 1+3)$; $(5+4, 1+6)$ or $(7, 4)$; $(9, 7)$.

Again, if x changes by $2t$, y changes by $3t$.

$\therefore$ $x=5+2t$, $y=1+3t$ are general expressions for x and y.

These are whole numbers if t is a whole number, but for all values of t they satisfy the equation.

Example IV. Find the equation of the line given by $x=1-5t$, $y=-2+3t$.

Solution. Eliminate t by multiplying the first equation by 3 and the second by 5 and then adding the equations.

Adding $3x=3-15t$ to $5y=-10+15t$ we get $3x+5y=-7$.

This is the required equation.

[The letter t which occurs in Examples III, IV, and which may be given any value whatever, is called a *parameter*. The equations containing t are called the *parametric* equations of the line.]

Example V. Discuss the solution of the equations

$$2x+3y-5=0,$$
$$4x+ay+7=0,$$

for varying values of a.

The solution is $$\frac{x}{21+5a}=\frac{-y}{14+20}=\frac{1}{2a-12}.$$

$\therefore$ for all values of a except $a=6$ we have

$$x=\frac{21+5a}{2a-12} \quad \text{and} \quad y=\frac{-34}{2a-12}, \qquad \text{.....................(i)}$$

but if $a=6$ there is no solution.

Consider the graphs :

$2x+3y-5=0$ is the line of gradient $-\frac{2}{3}$ through the point $y=0$, $x=\frac{5}{2}$;

$4x+ay+7=0$ is the line of gradient $-\frac{a}{4}$ through the point $y=0$, $x=-\frac{7}{4}$.

Through this point, *one* line can be drawn parallel to the first line; this is given by $a=6$ so that for $a=6$, the lines do not meet and there is no solution.

Innumerable other lines can be drawn through $(-\frac{7}{4}, 0)$, all of which will meet the first line and the point of meeting is given by (i).

Examples 8

1. In the following cases a linear equation connects x and y. Find the gradient of the line obtained by plotting the points; also find the linear equation :

(i)

$x=$	1	2	3
$y=$	3	5	7

(ii)

$x=$	-2	0	4
$y=$	3	6	12

2. Select from the following two tables the one which gives points in a straight line, and find the equation of that line.

(i)

x	1	4	6
y	1	3	5

(ii)

x	1	4	10
y	1	3	7

3. A is (4, 2), B is (4, 4), O is the origin (0, 0). Find the gradient of the median drawn from O of the triangle OAB; find also the gradient of the median drawn from B.

4. If the equation $ax+by+c=0$ is satisfied by $(\frac{1}{5}, -1)$ and by $(2, \frac{1}{2})$ prove that $b=\dfrac{22c}{19}$ and find a.

5. Find 3 pairs of values of x and y satisfying the equation in each of the following; also find general expressions for x and y.

(i) $4x-3y=7$; (ii) $2x+y=5$; (iii) $7x-4y=15$.

6. Find the equations whose solutions in terms of parameter t are

(i) $x=\frac{1}{4}+t,\ y=\frac{3}{4}-7t$; (ii) $x=5-2t,\ y=-6-3t$.

7. Find the equations of the lines (i) and (ii)

(i) from the table

x	a	c
y	b	d

;

(ii) from the general parametric equations $x=a+bt$, $y=c+dt$, from which t has to be eliminated.

8. The parametric equations

$\left.\begin{matrix} x=4+3t \\ y=7+2t \end{matrix}\right\}$ give the same line as the equations $\left\{\begin{matrix} x=1+3t' \\ y=5+2t' \end{matrix}\right.$.

Explain this statement and give the equation of the line.

9. Show that the equations $x=2+r\cos\theta$, $y=-1+r\sin\theta$ give the line of gradient $\tan\theta$ through $x=2$, $y=-1$. Find θ if the line passes through $x=5$, $y=3$, and the value of r needed to give this point.

10. Show that $(y-y_1)(x_2-x_1)=(x-x_1)(y_2-y_1)$ is the equation of the line joining (x_1, y_1) to (x_2, y_2).

Also show that $(y+y_1)(x_2-x_1)=(x+x_1)(y_2-y_1)$ is the equation of the line joining $(-x_1, -y_1)$ to $(-x_2, -y_2)$.

11. Discuss the solution of the equations $5x-y-3=0$, $ax+2y+4=0$ for varying values of a.

Show that it is possible to choose a value for a so that the lines given by the equations meet at the point $(1, 2)$.

12. Show that the equation $(3+2\lambda)x+(2-5\lambda)y-19=0$ is satisfied by $x=5$, $y=2$ whatever the value of λ.

Also show that with a particular value of λ the equation has no solution common to itself and $3x-17y+2=0$.

Using Equal Fractions

If $\frac{a}{b}=\frac{c}{d}=k$ so that $a=bk$ and $c=dk$, then whatever the values of p and q each fraction $=\frac{pa+qc}{pb+qd}$; for

$$\frac{pa+qc}{pb+qd}=\frac{pbk+qdk}{pb+qd}=k.$$

In exactly the same way, if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$, each of these fractions is equal to

$$\frac{pa+qc+le+mg}{pb+qd+lf+mh}.$$

The first result may be used to find where a line through the origin meets another line.

Thus if $\frac{x}{7}=\frac{y}{2}$ and $3x+5y=1$,

$$\text{each fraction}=\frac{3x+5y}{3\cdot 7+5\cdot 2}=\frac{1}{31},$$

$$\text{so that } x=\frac{7}{31} \text{ and } y=\frac{2}{31}.$$

This is perhaps shorter and certainly more elegant than writing

$y=\frac{2x}{7}$; $\therefore\ 3x+5\cdot\frac{2x}{7}=1$; $\therefore\ 21x+10x=7$.

$$\text{Whence } x=\frac{7}{31} \quad\text{and}\quad y=\frac{2}{7}x=\frac{2}{31}.$$

Example I. If $\frac{x-y}{5}=\frac{x+y}{11}=\frac{2x-y+z}{14}$ find $x:y:z$ and the value of $\frac{3x+y-5z}{x-2y+7z}$.

If $\qquad \frac{x-y}{5}=\frac{x+y}{11}$ each $=\frac{(x-y)+(x+y)}{5+11}=\frac{(x+y)-(x-y)}{11-5}$,

that is $\qquad =\frac{2x}{16}=\frac{2y}{6}.$

$$\therefore\ \frac{x}{8}=\frac{y}{3}=\frac{(2x-y+z)-2x+y}{14-16+3}=\frac{z}{1}.$$

$$\therefore\ x:y:z=8:3:1.$$

Writing $x=8k$, then $y=3k$ and $z=k$;

$$\therefore\ \frac{3x+y-5z}{x-2y+7z}=\frac{24k+3k-5k}{8k-6k+7k}=\frac{22k}{9k}=\frac{22}{9}.$$

Examples 9

1. If x and y are in a constant ratio, complete the following table:

x	10	30		95	125	
y	14		84			147

2. Complete the following :

(i) $\frac{x}{9}=\frac{y}{2}=\frac{x+y}{\quad}=\frac{x-y}{\quad}=\frac{4x+3y}{\quad}$;

(ii) $\frac{a+b-c}{10}=\frac{b+c-a}{4}=\frac{c+a-b}{14}=\frac{a+b+c}{\quad}=\frac{2a}{\quad}=\frac{2b}{\quad}=\frac{2c}{\quad}$;

(iii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{2a+b}{b-a}=\frac{\quad}{d-c}=\frac{2e+f}{\quad}$.

3. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$ show that

(i) $\frac{pa^2+qc^2+re^2}{pb^2+qd^2+rf^2}=k^2$; (ii) $\frac{pac+qce+rea}{pbd+qdf+rfb}=k^2$;

(iii) $\frac{a^2+b^2}{a+b}+\frac{c^2+d^2}{c+d}+\frac{e^2+f^2}{e+f}=\frac{(a+c+e)^2+(b+d+f)^2}{(a+c+e)+(b+d+f)}$.

4. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$ show that $x+y+z=0$ and $ax+by+cz=0$.

5. Find $x:y:z$ if $\frac{3x-2y+4z}{43}=\frac{x+3y-z}{38}=\frac{2x+y-3z}{27}$.

6. Show that the line $\frac{x}{4}=\frac{y}{5}$ meets the line $2y-x=24$ at a point twice as far from the origin as where it meets $x+y=18$.

7. Find x, y and z if $x:y:z=3:5:7$ and $2x-y+4z=58$.

8. If $x:y=4:3$ find the value of (i) $\frac{7x-2y}{5x+4y}$; (ii) $\frac{x^2-y^2}{x^2+y^2}$.

9. If $\frac{a}{b}=\frac{b}{c}$ show

(i) $\frac{a}{a+b}=\frac{b}{b+c}$; (ii) $\frac{a-b}{b}=\frac{b-c}{c}$;

(iii) $\frac{pa+qb}{ra+sb}=\frac{pb+qc}{rb+sc}$; (iv) $\frac{a^2}{a^2+b^2}=\frac{a^2-b^2}{a^2-c^2}$.

10. If $x:y:1=a(1-t^2):2bt:(1+t^2)$, show $b^2x^2+a^2y^2=a^2b^2$.

Miscellaneous Examples 10 [a, b, c, d are constants.]

1. Solve the equations :

(i) $\frac{1}{2}x-\frac{1}{3}x=12$; (ii) $\frac{1}{3}x-\frac{1}{4}x=a$; (iii) $\frac{x}{b}-\frac{x}{c}=d$.

2. If $v_1=u+at$ and $v_2=2u-at$ find t in terms of u and a so that $v_1=v_2$. Also find t if it is given that $v_1=2v_2$.

3. Solve the pairs of equations (i) $\frac{1}{2}x+y=7$, $x+\frac{1}{3}y=9$; (ii) $\frac{1}{3}x+\frac{1}{2}y=5$, $x+y=12$.

4. (i) Solve $a(x-a)+b(x-b)+c(x-c)=0$;
   (ii) Show that any value of x will satisfy the equation
$$(a-b)(x-c)+(b-c)(x-a)+(c-a)(x-b)=0;$$
in other words, prove the equation to be an *identity*.

5. If $a=b=c$ then $a=b$ and $a=c$; use this fact to solve the equations
$$2x+y+1=x-3y-6=4y+9.$$

6. Solve for x: (i) $3(5+x)=4(7x+10)$; (ii) $a(b+x)=c(2ax+3b)$.

7. If $y=ax+b$ find a and b from the pairs of values

(i)

x	-3	2
y	-4	11

; (ii)

x	-3	5
y	1	5

8. From $\dfrac{x}{d}-\dfrac{x-cd}{2d}=a$ prove that $x=(2a-c)d$.

9. If $s=\frac{1}{2}(u+v)t$:
   (i) find u if $s=300$, $t=12$, $v=35$;
   (ii) find t if $s=-22$, $u=16$, $v=-27$.

10. Of two numbers twice the larger is less by six than five times the smaller, while twice the smaller is less than the larger by one; find the numbers.

11. Solve the equations
   (i) $9-\frac{1}{2}(x-3)=\frac{1}{4}(x+3)$;
   (ii) $\frac{1}{4}(x+1)-\frac{1}{6}(x-1)=1$.

12. Find the point of intersection of each of the following pairs of lines:
   (i) $y=2x+5$, $y=5x-1$; (ii) $4x+3y-7=0$, $6x-y+6=0$; (iii) $\frac{1}{2}x+\frac{1}{4}y=1$, $2y-3x=2$.

13. Divide £345 between A, B and C so that B and C each get £15 less than twice what A gets.

14. If $s=ut+\frac{1}{2}at^2$:
   (i) find a given $s=164$, $u=35$, $t=4$;
   (ii) find u given $s=28$, $t=3\frac{1}{2}$, $a=16$.

15. Evaluate the determinants:
   (i) $\begin{vmatrix} 5 & 2 \\ 6 & 1 \end{vmatrix}$; (ii) $\begin{vmatrix} 9 & 7 \\ -4 & 4 \end{vmatrix}$; (iii) $\begin{vmatrix} a+b & b \\ a-b & 2a-b \end{vmatrix}$.

16. Find the equation of the line joining the points (1, 7), (3, 4). What is the gradient of this line?
   Show that the point (5, 0) lies on the line through (1, 6) of the same gradient.

17. Find two pairs of numbers such that they differ by 40 and one of them is double the other.

18. At what point does the line $\frac{x}{a}+\frac{y}{b}=1$ intersect the line $2ay-bx=a^2$?

19. Solve the equations:
$$x+y+z=29,$$
$$3x-2y=24,$$
$$x-8=2(17-z).$$

20. Prove that

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}+b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$
$$+a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+b_2\begin{vmatrix} c_1 & a_1 \\ c_3 & a_3 \end{vmatrix}+c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}=0.$$

CHAPTER II

FACTORS AND QUADRATICS

Factors

Revision. Trinomials : Difference of Squares

Example I. Factorise (i) $2x^2 - 5x - 3$; (ii) $4a^2 - 9b^2$.

(i) $2x^2 - 5x - 3 = (2x + 1)(x - 3)$
$$\begin{array}{|c} 2x \quad + \quad 1 \\ \\ x \quad - \quad 3 \end{array}$$

(ii) $$\begin{aligned} 4a^2 - 9b^2 &= (2a)^2 - (3b)^2 \\ &= (2a + 3b)(2a - 3b). \end{aligned}$$

Examples 11. Factorise :

1. $x^2 - 11x + 30$.
2. $x^2 - 11x - 12$.
3. $x^2 - 11x - 210$.
4. $2x^2 + 5x + 2$.
5. $3x^2 + x - 14$.
6. $3x^2 + 2x - 5$.
7. $144a^2 - 169b^2$.
8. $1 - 225x^2$.
9. $3 - 5y^2 - 2y^4$.
10. $2x^2 + 9x - 35$.
11. $3y^2 - 4y - 4$.
12. $15x + 2 - 8x^2$.
13. $5s^2 - 26st + 5t^2$.
14. $2 - 19y + 35y^2$.
15. $4 - 8x - 21x^2$.
16. $(a + b)^2 - c^2$.
17. $15z^2 - az - 2a^2$.
18. $72 - 17x - 72x^2$.

Factors involving square roots

Just as the equation $x^2 = 9$ giving $x = +3$ or -3 is associated with the factors of $x^2 - 9$, namely $(x - 3)(x + 3)$, so the equation $x^2 = 5$ giving $x = \sqrt{5}$ or $-\sqrt{5}$ is associated with the factors of $x^2 - 5$, namely $(x - \sqrt{5})(x + \sqrt{5})$. These factors are of *a new type, not hitherto thought of as factors.*

19. Factorise : (i) $x^2 - 3$; (ii) $(2x)^2 - 11$; (iii) $x^2 - 5y^2$.
20. Factorise : (i) $(x - 1)^2 - 3$; (ii) $(2x - 5)^2 - 7$.

Completing the square

Example II. What must be added in the following cases to complete the square? Write the result as a square : (i) $x^2 - 32x$; (ii) $x^2 + 9x$; (iii) $9x^2 - 24x$.

[The process of *completing the square* depends on the identities

$$(x + p)^2 \equiv x^2 + 2px + p^2 \quad \text{and} \quad (x - p)^2 \equiv x^2 - 2px + p^2.]$$

(i) Add the square of half the coefficient of x, i.e. add 16^2 or 256.
Result : $x^2 - 32x + 256 = (x - 16)^2$.

(ii) Similarly, result is $x^2+9x+(\frac{9}{2})^2=(x+\frac{9}{2})^2$.

(iii) Since $9x^2=(3x)^2$, take $24x$ as $8 \cdot 3x$ and add 4^2 or 16.

Result : $9x^2-24x+16=(3x-4)^2$.

Examples 12

What must be added in the following cases to complete the square? Write the result as a square.

1. (i) x^2+14x ; (ii) x^2-13x ; (iii) $16x^2-40x$.

2. (i) y^2+5y ; (ii) a^2-11ab ; (iii) $9x^2-60xz$.

Quadratic Equations

Quadratic equations, those involving x^2 and having two answers, are usually written in the form $ax^2+bx+c=0$, a quadratic expression equal to zero.

If the quadratic expression has simple factors, it is best to use these factors to get the solutions, or *roots* as they are often called.

Thus $2x^2-11x+4=0$ can be written $(x-2)(2x-7)=0$;

$$\therefore \text{ either } x-2=0 \text{ or } 2x-7=0.$$

So the equation is satisfied if $x=2$ and also if $x=3\frac{1}{2}$.

In such a case the roots are *rational* numbers, that is integers or the ratio of two integers.

Any number which cannot be expressed as the ratio of two integers is called an *irrational* number, and quadratic equations often have irrational numbers as roots, numbers involving *surds* such as $\sqrt{3}$ or $\sqrt{7}$. Approximate values of these are given in the square-root tables.*

In this case the equations are best solved by using the formula or by " completing the square ", which is the process by which the formula is proved.

Either this process or the formula can be used if the roots are rational, so do not labour to find factors if they do not come easily.

The general solution

Given $ax^2+bx+c=0$.

Divide by a and transpose the c/a :

$$x^2+\frac{b}{a}x \qquad\qquad = -\frac{c}{a}.$$

* See p. 43 for a discussion about surds and some methods which simplify work with them.

Complete the square on the left by adding $\left(\frac{b}{2a}\right)^2$ to each side:

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

Take the square root of both sides, not forgetting that either $+d$ or $-d$ has d^2 as its square.

$$x+\frac{b}{2a}=\pm\frac{\surd(b^2-4ac)}{2a};$$

$$\therefore \textbf{if } \mathbf{ax^2+bx+c=0}, \quad \mathbf{x=\frac{-b\pm\surd(b^2-4ac)}{2a}}. \quad \ldots\ldots\ldots\text{(I)}$$

This is the formula for the solutions.

Notice that the sum of the roots is $-b/a$; this is a valuable check.

There is another version of this formula which is recommended to technical students * who are liable to meet equations with awkward coefficients. It will usually be necessary to reduce the equation first to the form in which the coefficient of x^2 is 1.

Write $x^2+\frac{b}{a}x+\frac{c}{a}=0$ as $x^2+2px+q=0$; (notice the 2).

The square is completed by adding p^2 so as to get

$$x^2+2px+p^2=p^2-q,$$

and we get
$$x+p=\pm\surd(p^2-q);$$

$$\therefore \textbf{ if } \mathbf{x^2+2px+q=0}, \quad \mathbf{x=-p\pm\surd(p^2-q)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

Here the sum of the roots is $-2p$.

Either (I) or (II) should be selected and memorised.

No attempt should be made to learn both.

Example I. Solve the equations (i) $2x^2+5x+2=0$; $x^2+48=26x$.

(i) $2x^2+5x+2=0$;

$\therefore (2x+1)(x+2)=0$;

$\therefore$ either $2x+1=0$

or $x+2=0$;

$\therefore x=-\frac{1}{2}$ or -2.

(ii) $x^2+48=26x$;

$\therefore x^2-26x+48=0$;

$\therefore (x-24)(x-2)=0$;

$\therefore$ either $x-24=0$ or $x-2=0$.

$\therefore x=24$ or 2.

* M.A. report: *The Teaching of Mathematics in Technical Colleges*, p. 10.

Example II. Solve the equations

(i) $3x^2+12x-10=0$; (ii) $\frac{1}{2}\pi x^2+10x=15\cdot2$

(i) $3x^2+12x-10=0$;

$\therefore\ x=\dfrac{-12\pm\sqrt{144+120}}{6}$

$=\dfrac{-12\pm\sqrt{264}}{6}$

[using 4-figure tables]

$=\dfrac{-12\pm16\cdot25}{6}$

$=\dfrac{4\cdot25}{6}$ or $\dfrac{-28\cdot25}{6}$

$=\cdot71$ or $-4\cdot71$ to 2 d.p.

Check: Sum of roots $=-4$.

Note. (i) reduces to

$x^2+4x=3\cdot33$.

Completing the square:

$x^2+4x+4=7\cdot33$,

$x+2=\pm\sqrt{7\cdot33}$

$=\pm2\cdot71$,

$x=\cdot71$ or $-4\cdot71$,

or at once by the second formula,

$x=-2\pm\sqrt{4+3\cdot33}$.

(ii) $\frac{1}{2}\pi x^2+10x=15\cdot2$.

Taking $\pi=22/7$ and mult. by 7,

$11x^2+70x-106\cdot4=0$.

Working to 2 d.p. this is

$x^2+6\cdot36x-9\cdot67=0$.

$\therefore\ x=-3\cdot18\pm\sqrt{3\cdot18^2+9\cdot67}$

$=-3\cdot18\pm\sqrt{10\cdot11+9\cdot67}$

$=-3\cdot18\pm\sqrt{19\cdot78}$

$=-3\cdot18\pm\ 4\cdot45$

$=1\cdot27$ or $-7\cdot63$ to 2 d.p.

Check: Sum of roots $=-6\cdot36$.

Examples 13

1. Solve, using factors:

(i) $2x^2+28=15x$; (ii) $2x^2+9x-35=0$;
(iii) $3x^2=10x-8$; (iv) $x(x+1)=8x$.

2. Solve, using the formula or by completing the square:

(i) $3x^2-4x-2=0$; (ii) $x(x-8)=1$;
(iii) $9x(x+1)=3x+1$; (iv) $4x(x+5)=47$.

Solve the following equations:

3. (i) $x^2+2x=6$; (ii) $x^2+3x=7/4$.
4. (i) $x(x+1)-(x-1)(x+2)=2$; (ii) $x(x+1)+(x-1)(x+2)=2$.
5. (i) $\pi x^2+6x=30$; (ii) $x(x+1)+2\pi x=30$.
6. (i) $7x^2+9x=4$; (ii) $6x^2+7x-20=0$.
7. (i) $2x^2-11x+9=0$; (ii) $2x^2-11x+3=0$.
8. (i) $x^2+62\cdot2x+248\cdot6=0$; (ii) $3x^2+50\cdot4x+819\cdot6=0$.
9. From the formula $s=ut-16t^2$ find t (i) if $s=6$, $u=50$; (ii) $s=100$, $u=160$.

10. Given $x^2+y^2=6x+4y$, find (i) the values of y if $x=1$; (ii) the values of x if $y=1$.

11. Find 2 values of the ratio of x to y from:

    (i) $6x^2-29xy+28y^2=0$; (ii) $x^2+2xy-8y^2=0$.

12. Solve for a/b from:

    (i) $2a^2-9ab-5b^2=0$; (ii) $12a^2-17ab=7b^2$.

13. Solve for z (i) $z(z-3)=40$; (ii) $z(z+5)=1$.

Solve the equations:

14. (i) $\dfrac{x+4}{x-2}=\dfrac{2x-7}{3(x-8)}$; (ii) $\dfrac{2}{x}=\dfrac{3+x}{x^2-5}$.

15. (i) $\dfrac{x+3}{9}=\dfrac{51}{x-7}$; (ii) $\dfrac{3x+7}{x^2}=\dfrac{26}{3x+2}$.

16. (i) $x(x+1)+(x-1)(x+2)=-2$; (ii) $x(x+1)+(x-1)(x+2)=0$.

17. The sum of the squares of two consecutive numbers is 2813; find them.

18. The sum of the squares of two consecutive even numbers is 7444; find them.

19. The sum of the squares of two consecutive odd numbers is 3202; find them.

20. The sum of the reciprocals of two consecutive even numbers is 9/40; find them.

21. The difference of the reciprocals of two numbers which differ by 3 is 1/60; find the numbers.

22. The length of a rectangle is half as much again as its breadth. If the area is 1176 sq. ft., what is the length?

The Discriminant

The expression b^2-4ac which occurs in formula (I) for the solution of a quadratic equation is called the *discriminant*, for it decides the nature of the roots.

Let $D\equiv b^2-4ac$; the possible cases are:

(i) $D>0$: there are two distinct roots, which are rational if D is a perfect square and irrational if D is not a perfect square. Corresponding to each root there is a factor of the quadratic expression, as explained below.

(ii) $D=0$: the same value for the root is obtained whether the *plus* or *minus* sign is used; i.e. the equation has two equal roots, and the two factors of the quadratic expression are identical.

(iii) $D<0$: there are no roots, for a negative number has no square root since all squares are positive.*

The discriminant for $x^2+2px+q=0$ is $4(p^2-q)$, but (p^2-q) may be considered instead.

Factors

If D is positive, factors may always be found for ax^2+bx+c, though unless D is a perfect square, the factors are of the new type, not hitherto thought of as factors, since they involve irrational numbers.

$$ax^2+bx+c=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right\}=a\left\{x+\frac{b}{2a}+\frac{\sqrt{D}}{2a}\right\}\left\{x+\frac{b}{2a}-\frac{\sqrt{D}}{2a}\right\}.$$

For example, $3x^2-5x+1$ for which $D=13$ has the factors

$$3\left\{x-\frac{5}{6}+\frac{\sqrt{13}}{6}\right\}\left\{x-\frac{5}{6}-\frac{\sqrt{13}}{6}\right\} \text{ or } \frac{1}{12}\{6x-5+\sqrt{13}\}\{6x-5-\sqrt{13}\}.$$

These are factors of the new type.

If D is a perfect square the factors *can* be found in this way. Thus $3x^2-5x+2$, for which $D=1$ has the factors

$$3\left(x-\frac{5}{6}+\frac{1}{6}\right)\left(x-\frac{5}{6}-\frac{1}{6}\right) \text{ or } \frac{1}{12}(6x-4)(6x-6)=(3x-2)(x-1),$$

but it is far easier to find these factors by inspection.

Sum and Product of Roots

Suppose that α and β are the roots of $ax^2+bx+c=0$, that is of

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Then
$$x^2+\frac{b}{a}x+\frac{c}{a}=(x-\alpha)(x-\beta)$$
$$=x^2-(\alpha+\beta)x+\alpha\beta;$$
$$\therefore\ \alpha+\beta=-\frac{b}{a},\quad \alpha\beta=\frac{c}{a}. \quad \text{..................(III)}$$

These results are usually called the formulae for the *sum of the roots* and the *product of the roots* of the quadratic equation $ax^2+bx+c=0$.

Many results can be deduced by using these formulae (see Examples 10 to 14 below).

* The desire to say that in this case also there are two roots has led to a most important development in the idea of "number", for which see Chap. XI.

Examples 14

1. Find the discriminant for each of the following equations:

 (i) $7x^2+9x+2=0$; (ii) $7x^2+9x-3=0$.
 (iii) $6x^2+7bx-20b^2=0$; (iv) $9a^2x^2+36abx+35b^2=0$.

 Which of these equations will have rational roots?

2. Show that the discriminant of the equation $(x-4)^2+16x=0$ is zero and solve the equation.

3. By examining the discriminant state the character of the roots of the following equations:

 (i) $2x^2-19x+42=0$; (ii) $2x^2-10x+11=0$;
 (iii) $8x^2+12x+29=0$; (iv) $4x^2-44x+121=0$.

4. Find the value of p for which each of the following has the discriminant equal to zero. Why is the second expression not then a perfect square?

 (i) $4x^2-2px+49$; (ii) $18x^2+2px+32$.

5. Give the factors (containing surds) of

 (i) x^2-8x+9; (ii) x^2+4x-1.

6. Show that there are no factors for

 (i) $x^2+2x+27$; (ii) $3x^2+10x+12$.

7. Which of the following can be factorised? Find the factors.

 (i) x^2+x-1; (ii) x^2+x+1;
 (iii) x^2-x+1; (iv) x^2-x-1.

8. In the equation $x(x+1)+(x-1)(x+2)=a$ show that if $2a=-5$ there are equal roots and that if $2a<-5$ there are no roots.

9. Which of the following equations has roots? Solve it and show that the other has no roots.

 (i) $x^2+20x+50=0$; (ii) $x^2+20x+150=0$.

[*Calling the roots of the quadratic equation* α *and* β.]

10. For each of the following equations find the values of $\alpha+\beta$, $\alpha\beta$, $\alpha^2+\beta^2$ and $(\alpha-\beta)^2$. [Note that $\alpha^2+\beta^2\equiv(\alpha+\beta)^2-2\alpha\beta$.]

 (i) $x^2+x-56=0$; (ii) $x^2-4x+2=0$;
 (iii) $2x^2+7x+4=0$; (iv) $x^2+x-3=0$.

11. Find the value of $\alpha^2+\beta^2$ and of $(\alpha-\beta)^2$ for the equations

 (i) $ax^2-3bx+4c=0$; (ii) $4a^2x^2+36abx+81b^2=0$.

12. If $x^2+2px+q=0$ has roots α, β, show that $\alpha+\beta=-2p$, $\alpha\beta=q$, and express in terms of p and q

 (i) $\alpha^2+\beta^2$; (ii) $\alpha^4+\alpha^2\beta^2+\beta^4$.

 [Hint. The second expression is $(\alpha^2+\beta^2)^2-\alpha^2\beta^2$.]

13. If $1+px+qx^2\equiv(1-\alpha x)(1-\beta x)$ express in terms of p and q

 (i) $\alpha+\beta$; (ii) $\alpha\beta$; (iii) $\alpha^2-\alpha\beta+\beta^2$; (iv) $\alpha^4+\beta^4$.

14. If α, β are the roots of $ax^2+bx+c=0$, prove that

$$\alpha^2+\beta^2=(b^2-2ac)/a^2 \text{ and } \alpha^4+\beta^4=(b^4-4ab^2c+2a^2c^2)/a^4.$$

15. If x_1, x_2 are the roots of $x^2-px+q=0$, show that

(i) $x_1^3+x_2^3=p^3-3pq$; (ii) $x_1^3-x_2^3=(p^2-q)\sqrt{(p^2-4q)}$.

16. If $x^2-px+q=0$ show that $x^n-px^{n-1}+qx^{n-2}=0$, and hence, if x_1 and x_2 are the roots, find A and B so that

$$x_1^n+x_2^n=A(x_1^{n-1}+x_2^{n-1})+B(x_1^{n-2}+x_2^{n-2}).$$

Use this to obtain $(x_1^4+x_2^4)$ in terms of p and q.

Equations from Roots

Example I. Form the equation whose roots are $\frac{2}{3}$ and $\frac{3}{4}$.

The equation is $(x-\frac{2}{3})(x-\frac{3}{4})=0$ which gives

$$(3x-2)(4x-3)=0 \text{ or } 12x^2-17x+6=0.$$

Example II. Form the equation whose roots are $3\pm\sqrt{17}$.

The equation is $(x-3-\sqrt{17})(x-3+\sqrt{17})=0$ or $(x-3)^2-17=0$

$$\text{i.e. } x^2-6x-8=0.$$

Examples 15

1. Form the equations whose roots are

(i) $2\frac{1}{2}$, $-3\frac{1}{3}$; (ii) $-6\pm\sqrt{23}$.

2. Find equations whose roots are

(i) $3a$, $2b$; (ii) $a\pm\sqrt{p}$.

3. What is the equation whose roots are the squares of the roots of $x^2-4x+2=0$? [See Examples 14, No. 10 (ii).]

4. If $x^2+px+q\equiv(x-\alpha)(x-\beta)$ form the equation whose roots are $1/\alpha$ and $1/\beta$.

5. With the data of No. 4 form the equations whose roots are (i) α^2 and β^2; (ii) α/β and β/α.

6. (i) Form the equation whose roots are α and 3α.

(ii) One of the roots of $x^2+px+q=0$ is three times the other root. Show $3p^2=16q$.

7. If α and β are the roots of $ax^2+bx+c=0$, show that

$$\left(\alpha+\frac{1}{\beta}\right)\left(\beta+\frac{1}{\alpha}\right)=(a+c)^2/ac,$$

and find the equation whose roots are $\alpha+\dfrac{1}{\beta}$ and $\beta+\dfrac{1}{\alpha}$.

8. Form the equation whose roots are each one more than the roots of $x^2+px+q=0$.

9. Solve the following equations, being given in each case that one root is the cube of the other; also find p in (iii).

(i) $x^2-68x+256=0$, (ii) $x^2-222x+1296=0$;

(iii) $625x^2+px+81=0$.

10. If α and β are the roots of $x^2 - px + q = 0$, form the equation whose roots are (i) 2α, 2β, (ii) $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, (iii) $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$, (iv) $\alpha + \beta$, $\alpha\beta$.

Functions and Graphs

When an expression involving x, for example $2x^2 - 5x + 7$, is chosen and the question is not to find when it is zero, but what values it takes as x is given different values, the expression is being considered as a *function* of x and the way the function varies as x varies is being examined.

The student knows that first-degree functions are called *linear* functions because their graphs are straight lines. Also he has presumably drawn the graph of x^2 (or of $y = x^2$) and knows that a curve, the curve of squares, is obtained.

That the graph of the more general quadratic function $ax^2 + bx + c$ is of a shape similar to that of x^2 will be seen if the following examples (Fig. 2) are worked.

It is not intended that a large part of each graph should be drawn; the diagrams below show the sort of thing wanted.

In each of these diagrams the curve is a *parabola*, with its line of symmetry (its *axis*) parallel to the y-axis. The point at which the curve changes from going down to going up (or vice versa) is called

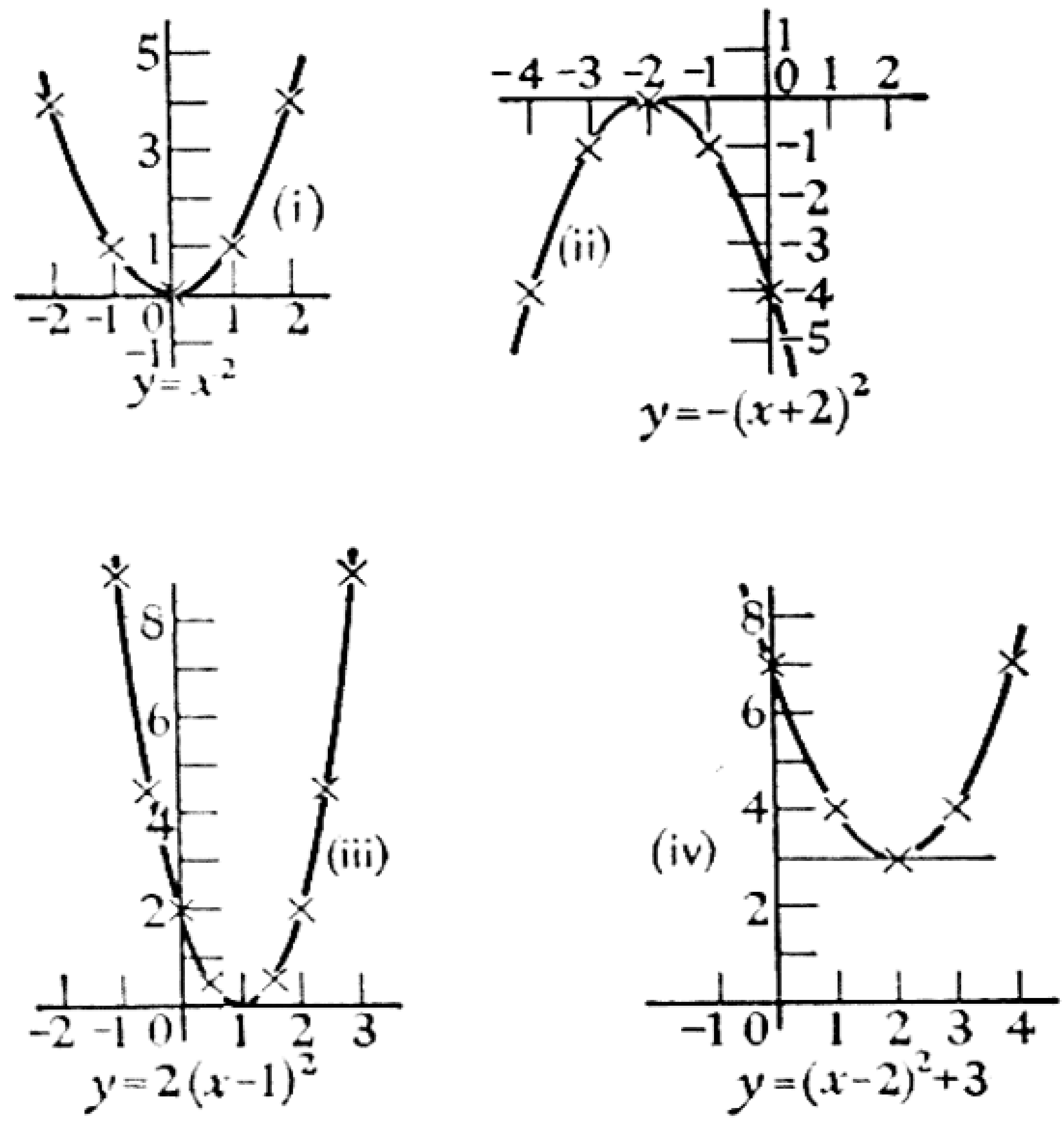

FIG. 2

the *vertex*. At this point the tangent is parallel to the x-axis, at right-angles to the axis of the parabola.

Examples 16

1. Draw the graphs for

$$\text{(i) } y = -x^2; \quad \text{(ii) } y = (x-3)^2; \quad \text{(iii) } y = (x-1)^2 + 2;$$
$$\text{(iv) } y = 2x^2; \quad \text{(v) } y = 3(x-2)^2 - 1, \quad \text{(vi) } y = -x^2 + 5.$$

[Several of these can be drawn on the same sheet of graph paper.]

2. Draw the graphs for $y = 2x^2$ and $y = 2(x - \frac{3}{2})^2 + \frac{1}{2}$ on the same paper and verify by measurement that they are congruent curves, the latter being the former moved a distance $1\frac{1}{2}$ units to the right and $\frac{1}{2}$ unit up.

Why do the graphs cut when $x = 5/6$?

3. To get the graph of $y = x^2 - 4x + 5$ rewrite the equation as

$$y = (x-2)^2 + 1.$$

Hence the graph is that which would be obtained by moving the graph of $y = x^2$ to the right 2 units and upwards 1 unit. Take new axes for X, Y through (2, 1), parallel to the original axes, and plot $Y = X^2$ on these axes (Fig. 3).

Deal similarly with

$$y = x^2 - 2x + \tfrac{1}{2}.$$

[As squares increase so much more rapidly than the numbers which are squared, it is often convenient to take the y-scale much smaller than the x-scale.]

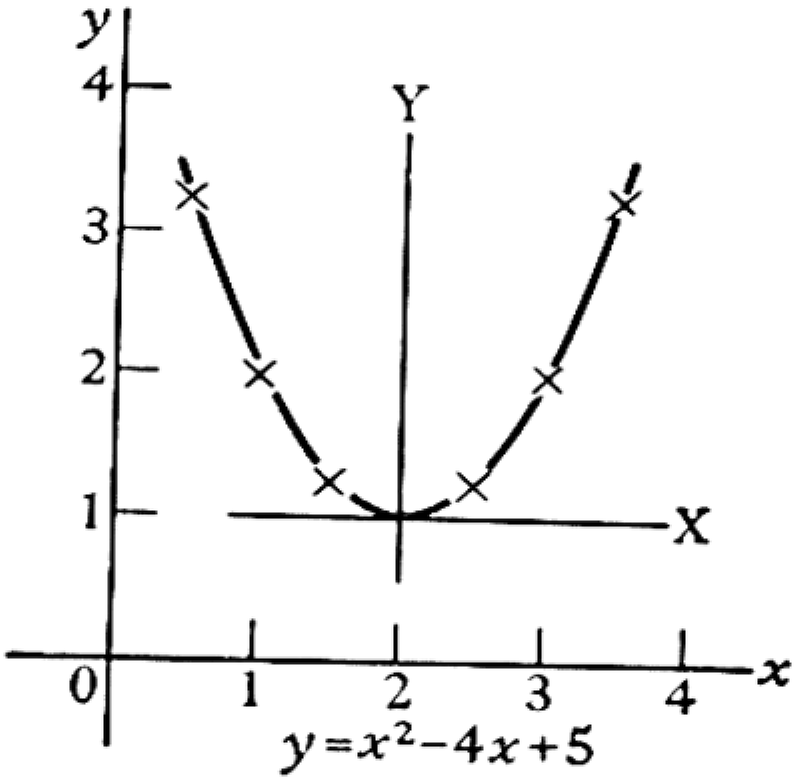

FIG. 3

4. Taking 1 inch as unit for x and $\frac{1}{5}$ inch as unit for y draw the graph of

$$y = 2x^2 + 4x, \text{ i.e. } y = 2(x+1)^2 - 2$$

for values of x from -3 to $+3$.

Read from the graph the negative value of x for which $y = x$.

5. If 1 inch is the unit on the x-axis and $\frac{1}{4}$ inch that on the y-axis, what is the equation of the line drawn to bisect the angle between the axes in the quadrant in which x and y are both positive.

Show that the graph of $y = 8(x-1)^2$ cuts this line where $x = 2$ and $x = \frac{1}{2}$.

The Graph of the General Quadratic Function

I. If $y = x^2 + 2px + q$, then $y = (x+p)^2 - (p^2 - q)$ or

$$y + (p^2 - q) = (x+p)^2.$$

Hence the graph of $y=x^2+2px+q$ may be obtained from the graph of $y=x^2$ by :

(i) moving it parallel to the x-axis through $-p$ so as to give the graph of $y=(x+p)^2$, followed by

(ii) moving it parallel to the y-axis through $-(p^2-q)$.

This means that the vertex of the parabolic graph is at $(-p, \ -\overline{p^2-q})$.

II. If $y=ax^2+bx+c$, then

$$y=a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}.$$

In this case the graph may be obtained by starting with the graph of $y=ax^2$, and then

(i) moving it parallel to the x-axis through $\frac{-b}{2a}$ so as to give the graph of $y=a\left(x+\frac{b}{2a}\right)^2$, followed by

(ii) moving it parallel to the y-axis through $-\frac{b^2-4ac}{4a}$.

Hence the vertex is at $\left(\frac{-b}{2a}, \ -\frac{b^2-4ac}{4a}\right)$.

Assuming a to be positive, the parabola will cut the x-axis if the vertex is below that line, i.e. if b^2-4ac is positive.

If b^2-4ac is *negative*, the equation has no roots and the graph of ax^2+bx+c must be entirely on one side of the x-axis, above if a is *positive* and below if a is *negative*.

If b^2-4ac is *positive*, the equation has roots ; the graph crosses the x-axis and ax^2+bx+c is sometimes positive and sometimes negative.

As stated on p. 24, the expression b^2-4ac is called the ***discriminant***.

Examples 17

1. Draw rough graphs of each of the following, indicating clearly the position of the vertex in each case.

   (i) $y=x^2-3x+7$; (ii) $y=2x^2-6x-3$; (iii) $y=5x^2+4x-11$;
   (iv) $y=6-4x-x^2$; (v) $y=5-3x-2x^2$; (vi) $y=15+4x-5x^2$.

2. Show that $3x^2-5x+4$ is positive for all values of x, and determine for what integral values of k the same is true of $3x^2-kx+4$.

3. Prove that there is no value of x for which $7x-1-14x^2$ is positive, and find the greatest value of c so that no value of x will make $c+8x-3x^2$ positive.

4. Prove that for all values of x the expression $2x^2-8x+10$ is never less than 2 and that the expression $1-4x-x^2$ is never greater than 5. Compare with the diagrams 2 and 5 above, explaining how they illustrate the results.
5. For what range of values of x is the expression $2x^2-14x+20$ negative? Prove that $2x^2-14x+25$ is never negative.
6. Prove that the greatest integral value of k for which $k+7x-x^2$ is never positive is -13. For what range of values of x is the expression positive when k is -12?

Maximum and Minimum Values

Note that $a-(x-b)^2$ is maximum if $x=b$ and that the maximum value is a; this is because the square term must be positive unless it is zero. Also for the same reason $(x-c)^2+d$ is minimum if $x=c$ and its minimum value is d.

Thus completing the square in x and writing a quadratic function as the sum or difference of a constant and a perfect square shows that the function has *either* a maximum value *or* a minimum value (but not both) and what that value is. The graph shows this also, the maximum or minimum being at the vertex of the parabola.

Example I. Decide whether x^2+3x-7 has a maximum or minimum value and find this value, and its position.

Solution. $x^2+3x-7=x^2+3x+(\frac{3}{2})^2-9\frac{1}{4}=(x+\frac{3}{2})^2-9\frac{1}{4}$.

$\therefore$ the function has a minimum value $-9\frac{1}{4}$ when $x=-\frac{3}{2}$.

Example II. Repeat Example I for the expression $8+12x-3x^2$.

Solution. $8+12x-3x^2=-3(x^2-4x)+8=-3(x^2-4x+4)+20$

$$=20-3(x-2)^2.$$

$\therefore$ the function has a maximum value 20 when $x=2$.

Examples 18. Arrange each of the following functions in the form $p(x+q)^2+r$; find the maximum or minimum value and the value of x giving this value:

1. x^2+4x+7.	2. x^2-8x+1.	3. $-x^2+3x$.
4. $3-x+x^2$.	5. $8-4x-4x^2$.	6. $2x^2-12x+23$.
7. $3x^2+6x-1$.	8. $x^2+2hx+k$.	9. $ax^2+2bx+c$.

10. Is the maximum value of y_1 greater or less than the minimum value of y_2 where

$$y_1=-2x^2+12x-13 \quad \text{and} \quad y_2=3x^2-12x+17?$$

11. Show that x^2+2x-3 is zero when $x=-3$ and $x=1$; that it is negative between these values and minimum half-way between them. For what range of values is $10x-21-x^2$ positive? where and what is its maximum value?

12. Find k so that the graph of $3x^2+4x+1-k$ may cut the x-axis. [If it does not, the equation $3x^2+4x+1=k$ is said to have "no real roots".]

The General Case

Note that since $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$ the decision as to a maximum or a minimum depends on the sign of a, minimum if a is positive, maximum if a is negative.

The vertex of the curve is the point where

$$x=-b/2a \text{ and } y=-(b^2-4ac)/4a,$$

and this point gives the position and value of the maximum or minimum, or, as it is called in either case, "*the turning point*".

The Functional Notation

It is often very convenient to use the notation $f(x)$ for the "function of x which is being considered", with the understanding that $f(p)$ will stand for the result of replacing x by p in the function.

Thus if $f(x)\equiv ax^2+bx+c$, then $f(p)=ap^2+bp+c$,

$$f(x+h)=a(x+h)^2+b(x+h)+c,$$

$$f(3)=9a+3b+c, \text{ and so on.}$$

$F(x)$, $\phi(x)$, etc. are used in the same way.

Examples 19

1. If $f(x)\equiv 3x^2+5x$ write down $f(x+a)$ and $f(x-a)$ and show that their sum is $6x^2+10x+6a^2$; also show that $f(x+a)-f(x-a)=12ax+10a$.
2. If $F(x)=8x^2-15x+2$ show that $F(1)=-5$ and find the value of $F(2)$.
3. If $\phi(x)=ax^2+bx+c$ show that $\phi(p)-\phi(-p)=2bp$.
4. Given that $f(x)\equiv x^3+3x^2-x-3$ show that three of $f(1)$, $f(-1)$, $f(3)$, $f(-3)$ are zero and find the value of the fourth.
5. If $f(x)\equiv px^2+qx+r$ find the value of $f(x+h)-f(x)$.
6. What powers of x can occur in $f(x)$ if $f(x)=f(-x)$, and what powers can occur if $f(-x)=-f(x)$?
7. If $f(x)=ax^2+bx+c$ find the value of $f(x)+f(-x)$ and of $f(x)-f(-x)$.
8. If $f(x)=\frac{1}{x}$ show that $f(x+a)+f(x-a)=\frac{2x}{x^2-a^2}$.
9. If $f(x, y)=3x^2+5xy+2y^2+7x+9y+5$, find the values of $$f(x, y)+f(-x, -y) \text{ and of } f(x, y)-f(-x, -y).$$

Gradient

At an early stage in the Calculus course it is shown that the gradient of a curve, which is the gradient of the tangent to the curve, is found by taking the limit of the gradient of a short chord. If the curve is $y=ax^2+bx+c$, the gradient is the value of $\frac{dy}{dx}$ at the point concerned and the process of differentiating gives

$$\frac{dy}{dx}=2ax+b.$$

Using the functional notation, which has just been explained, if

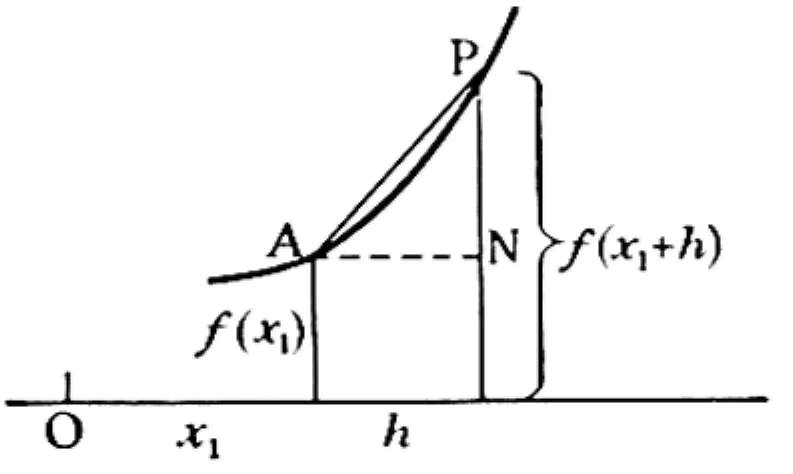

FIG. 4

the curve is $y=f(x)$ and if at A and P (Fig. 4) $x=x_1$ and x_1+h respectively, the gradient of the chord AP is

$$\frac{PN}{AN}=\frac{f(x_1+h)-f(x_1)}{h},$$

and the limit of this as $h \to 0$ gives the gradient at A, which is denoted by $f'(x_1)$, the value of $\frac{dy}{dx}$ in general being denoted by $f'(x)$.

The function $2ax+b$ may also be called the *derived function* or *derivative* of ax^2+bx+c.

Example I. For the curve $y=7x-3x^2$ find the gradient where $x=1$; also find the coordinates of the point where the gradient is zero and the gradient of the line joining this point to the point where $x=2$.

$\frac{dy}{dx}=7-6x$; the value of this when $x=1$ is 1.

$\frac{dy}{dx}=0$ if $x=\frac{7}{6}$; at this point $y=\frac{49}{6}-3\times\frac{49}{36}=\frac{49}{12}$.

Between this point and the point where $x=2$ and $y=14-12=2$, the x-step is $\frac{5}{6}$ and the y-step is $-\frac{25}{12}$.

So the gradient of the chord is $-\frac{5}{2}$.

Example II. For the curve $y=f(x)\equiv 7x^2-9x+2$ find the value of $f(x+h)$ and show that $f(x+h)-f(x)=h(14x-9)+7h^2$.

Verify that $\dfrac{dy}{dx}=$ the coefficient of h in $f(x+h)-f(x)$.

$$f(x+h)=7(x+h)^2-9(x+h)+2$$
$$=7x^2-9x+2+14xh-9h+7h^2;$$
$$\therefore\ f(x+h)-f(x)=h(14x-9)+7h^2.$$

Also by the rule, $\dfrac{dy}{dx}=14x-9$, and so is the coefficient of h in

$$f(x+h)-f(x).$$

Example III. If $f(x)=ax^2+bx+c$ show that

$$\frac{f(x_1+h)-f(x_1-h)}{2h}=2ax_1+b.$$

Interpret this result by completing the sentence " the tangent where $x=x_1$ is parallel to . . .".

$$f(x_1+h)-f(x_1-h)=a\{(x_1+h)^2-(x_1-h)^2\}+b\{(x_1+h)-(x_1-h)\}+c-c$$
$$=4ax_1h+2bh.$$

This proves the first result.

At $x=x_1$, $\dfrac{dy}{dx}$ has the value $2ax_1+b$, and $\dfrac{f(x_1+h)-f(x_1-h)}{2h}$ is the gradient of the chord joining the points where $x=x_1+h$ and $x=x_1-h$.

$\therefore$ the tangent where $x=x_1$ is parallel to this chord.

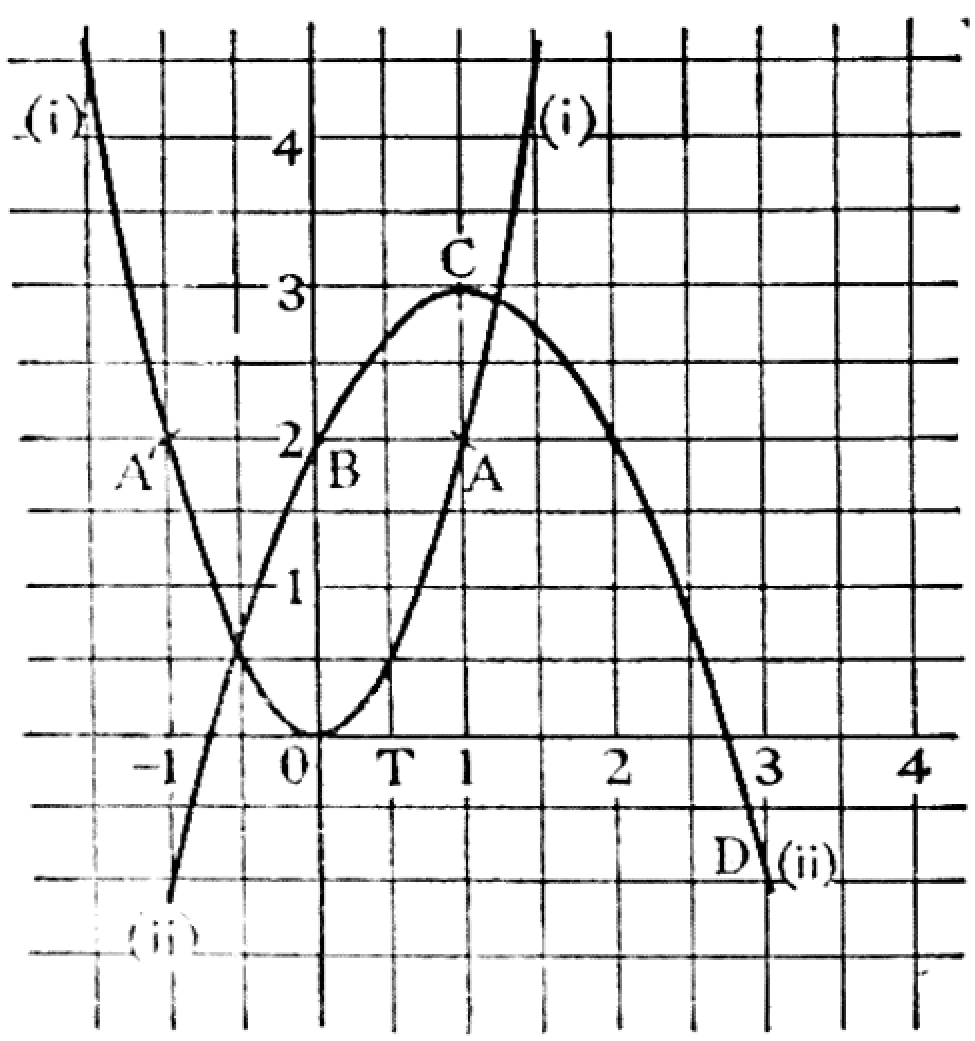

FIG. 5

Applications of Gradient

Fig. 5 shows the graphs of (i) $y=2x^2$ and (ii) $y=2+2x-x^2$.

For $2x^2$ and $2+2x-x^2$ the gradient-functions or derivatives are $4x$ and $2-2x$ respectively.

On curve (i) at the point A where $x=1$, the gradient $(4x)$ is 4.

Now if T is the point $(\frac{1}{2}, 0)$ the line TA rises 2 units while moving $\frac{1}{2}$ unit horizontally; thus its gradient is 4 and TA touches the curve at A.

On curve (ii) at B since $x=0$ the gradient is 2,
at C since $x=1$ the gradient is 0,
at D since $x=3$ the gradient is -4.

Notice first that the zero gradient at C corresponds to the *maximum* value of y; the tangent is horizontal.

A zero gradient and horizontal tangent may, however, indicate a *minimum* value of y as in curve (i) where the gradient $4x$ is zero when x is 0, that is at the origin.

Thus, **for a quadratic, a zero gradient means that the function is maximum or minimum.**

Again, since the gradient is 2 at B, the tangent could be got by joining B to the point $(-1, 0)$.

Examples 20

1. Copy curve (i), Fig. 5, and find the gradient at A', $(-1, 2)$. Draw the tangents at A and A' accurately and show both by drawing and by calculation that they meet at the point $(0, -2)$.
2. Show both by the use of the gradient and by the process of completing the square that the maximum value of $1+4x-x^2$ is 5 and is obtained by making x equal to 2.
3. Show that the minimum value of x^2-px is obtained by putting $x=\frac{1}{2}p$ and is $-\frac{1}{4}p^2$.
4. Given that
$$F(x)=4x^2-12x+27,$$
$$\phi(x)=9+12x-4x^2,$$
prove that the minimum value of $F(x)$ is equal to the maximum value of $\phi(x)$ and that the curves $y=F(x)$ and $y=\phi(x)$ touch each other.
5. For the curve $y=2+2x-x^2$ shown in Fig. 5 find the gradient at the point $(2, 2)$. Hence find where the tangent at this point cuts the axes. Show that the equation of this line is $y+2x=6$.
6. Find the minimum value of x^2+3x+3 and sketch its graph from $x=-3$ to $x=0$. Find the gradients of the graph at the points where it crosses $y=1$ and show that the tangents at these points are the diagonals of a square.

7. The height in feet of a body projected vertically from the ground with a velocity of 80 feet per second t sec. after projection is given by the formula $h = 80t - 16t^2$. Find the greatest height attained and the time taken to reach that height. What is the meaning of the derived function?

8. In Fig. 6 show that $AP^2 + PB^2 = 2x^2 - 10x + 30$ and find the value of x so that $AP^2 + PB^2$ should have its least value.

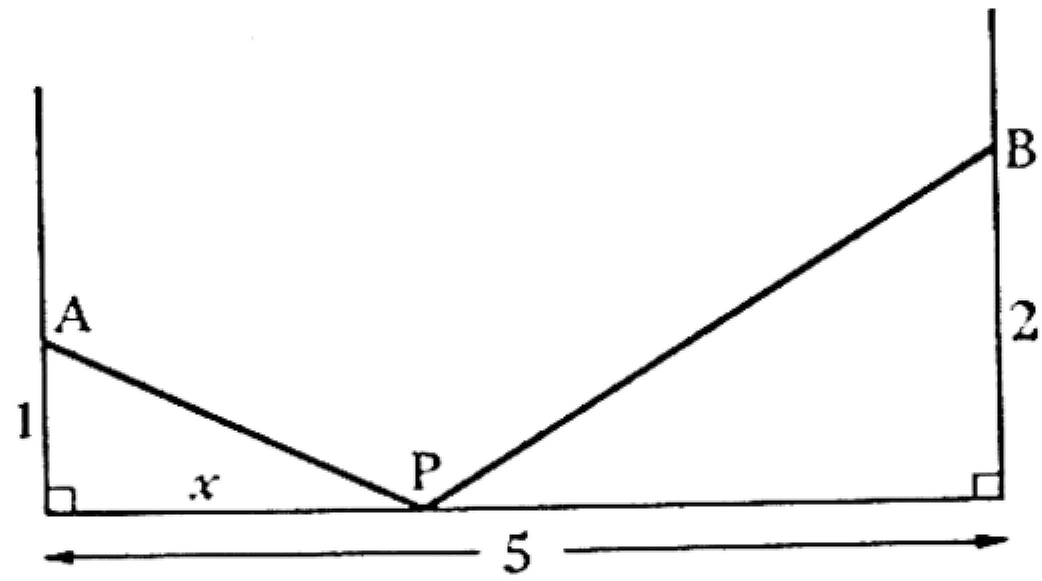

FIG. 6

9. A piece of string is 36 inches long and is placed so as to enclose a rectangle. Show that the area of the rectangle will be a maximum when it is a square.

10. A rectangular block of wood has a square section and its girth (the distance round the section) plus its length is 6 feet. Find the length when the product of girth and length is a maximim.

Factors of Other Types

So far the factors considered have as a rule been *binomials* (of two terms) whose product, the expression to be factorised, has been a *trinomial* (of three terms).

For *quadrinomials* or expressions of 4 terms, there is more variety in the types of factors possible.

Grouping

Example I. $a^3 + a^2b + ab^2 + b^3 = a^2(a+b) + b^2(a+b) = (a^2+b^2)(a+b)$.

Example II. $bx - ab + a^2 - ax = b(x-a) + a(a-x)$
$$= b(x-a) - a(x-a) = (b-a)(x-a).$$

The factors have two terms each. This grouping in pairs should always be tried first, unless the factors are seen at once.

Example III. $x^2 - y^2 + a(x-y) = (x+y)(x-y) + a(x-y)$
$$= (x+y+a)(x-y).$$

One factor has three terms, the other two.

Difference of squares

Example IV. $y^2+z^2-4x^2-2yz$

$$=(y^2-2yz+z^2)-4x^2=(y-z)^2-(2x)^2=(y-z+2x)(y-z-2x).$$

Example V. $a^2-b^2-c^2+2bc$

$$=a^2-(b^2-2bc+c^2)=a^2-(b-c)^2=(a+b-c)(a-b+c).$$

Each factor has three terms.

Examples 21. Factorise :

1. $a^3-a^2b+ab^2-b^3$.
2. $prx^2-psxy-qsy^2+rqxy$.
3. $bx-ab-a^2+ax$.
4. $al^2-alm-ln+mn$.
5. $3x^2+6x+2y+xy$.
6. a^3+a^2b-a-b.
7. $a^2-b^2-2a+2b$.
8. $4x^2-y^2+4x-2y$.
9. x^2-xz-y^2-yz.
10. $x^2-b^2-c^2+2bc$.
11. $x^2+2bx-c^2+b^2$.
12. $a^2-c^2-2b(2a+c)+3b^2$.
13. $x^4-a^4-b^4+2a^2b^2$.
14. $4(a+b)^2-(a-b)^2$.
15. $y^4-16+x^2-2xy^2$.

Sum or Difference of Two Cubes

The multiplications below show that the sum of two cubes and the difference of two cubes can be factorised.

$$\begin{array}{llll}
a^2 & -ab & +b^2 & \\
 & a & +b & \\
\hline
a^3 & -a^2b & +ab^2 & \\
 & a^2b & -ab^2 & +b^3 \\
\hline
a^3 & & & +b^3
\end{array}
\qquad\qquad
\begin{array}{llll}
a^2 & +ab & +b^2 & \\
 & a & -b & \\
\hline
a^3 & +a^2b & +ab^2 & \\
 & -a^2b & -ab^2 & -b^3 \\
\hline
a^3 & & & -b^3
\end{array}$$

Neither a^2-ab+b^2 nor a^2+ab+b^2 has factors.

Examples 22. Using the results

$$\mathbf{a^3+b^3=(a+b)(a^2-ab+b^2)}$$

and $\mathbf{a^3-b^3=(a-b)(a^2+ab+b^2)}$

factorise the following :

1. (i) x^3+y^3 ; (ii) x^3-8y^3 ; (iii) x^3+27y^3.
2. (i) $8a^3+1$; (ii) $125a^3b^3+64c^3$; (iii) $7xy^3-7xz^3$.
3. (i) x^6+a^6 ; (ii) x^6-a^6 ; (iii) $4c^3x^3-32c^3y^3$.
4. (i) $(a+b)^3+(a-b)^3$; (ii) $(a+b)^3-(a-b)^3$.

Factorise :

5. $7x^2-34xy-5y^2$.
6. $3x^4-31x^2+56$.
7. $ac^2+bd^2+bc^2+ad^2$.
8. $ac^3+bd^3-bc^3-ad^3$.
9. $x^2+2bx-c^2+b^2$.
10. $x^2+2bc-c^2-b^2$.
11. Work out $(a+b)^3$ and $(a-b)^3$; hence factorise
    (i) $8a^3+12a^2b+6ab^2+b^3$; (ii) $8x^3-36x^2y+54xy^2-27y^3$.

12. Express as difference of two squares and factorise

$$\text{(i) } a^4+a^2b^2+b^4\,; \quad \text{(ii) } a^4-a^2b^2+b^4.$$

[Solution. (ii) $a^4-a^2b^2+b^4=a^4+2a^2b^2+b^4-3a^2b^2$
$=(a^2+b^2+\sqrt{3}\,.\,ab)(a^2+b^2-\sqrt{3}\,.\,ab)$].

13. Verify that

$$a^4+b^4\equiv(a^2+b^2-ab\sqrt{2})(a^2+b^2+ab\sqrt{2}).$$

14. Verify that

$$\text{(i) } a^6+b^6\equiv(a^3)^2+(b^3)^2\equiv(a^2+b^2)(a^4-a^2b^2+b^4)\,;$$
$$\text{(ii) } a^6-b^6\equiv(a^3)^2-(b^3)^2\equiv(a+b)(a-b)(a^4+a^2b^2+b^4).$$

Factorise each of the brackets containing 3 terms as in No. 12.

Note that though $a^2+b^2\equiv\{a+b-\sqrt{(2ab)}\}\{a+b+\sqrt{(2ab)}\}$ these cannot be regarded as factors, because of the $\sqrt{(ab)}$, though if it were only the 2 under the root sign they could be thought of as factors as in No. 13.

Simultaneous Quadratics

Given two equations in x and y, one equation linear and the other quadratic, a quadratic equation in one of the letters can be formed and the solution completed.

Example I. Solve

$$\left.\begin{aligned} 3x^2-2xy-3x+4y-6&=0, \quad\ldots\ldots\text{(i)}\\ 2x-y&=5. \quad\ldots\ldots\text{(ii)}\end{aligned}\right\}$$

From (ii) $y=2x-5$ and substitution in (i) gives

$$3x^2-2x(2x-5)-3x+4(2x-5)-6=0,$$
$$\text{i.e. } 3x^2-4x^2+10x-3x+8x-20-6=0,$$
$$\text{i.e. } -x^2+15x-26=0,$$
$$\text{and so } (-x+2)(x-13)=0\,;$$
$$\therefore \text{ either } x=2 \text{ or } x=13.$$

Substitution in (ii) gives the corresponding values of y as -1 and 21;

$$\therefore \left.\begin{aligned}x&=2\\ y&=-1\end{aligned}\right\} \quad\text{or}\quad \left.\begin{aligned}x&=13\\ y&=21\end{aligned}\right\}$$

These solutions can be checked in equation (i).

Example II. Solve

$$\left.\begin{aligned} x^2+y^2&=74, \quad\ldots\ldots\text{(i)}\\ 4x-3y+1&=0. \quad\ldots\ldots\text{(ii)}\end{aligned}\right\}$$

[Again we substitute from the linear equation into the quadratic.]

From (ii) $4x=3y-1$, $x=\frac{1}{4}(3y-1)$.

Substitute in (i) $\frac{1}{16}(3y-1)^2+y^2=74$;

$$\therefore\ 9y^2-6y+1+16y^2=1184,$$
$$25y^2-6y-1183=0.$$

[There are factors, but they are not easy to guess.]

By the formula $y=\frac{1}{50}\{6\pm\sqrt{36+118300}\}$
$=\frac{1}{50}\{6\pm\sqrt{118336}\}$.

The table gives $\sqrt{1183} = 34 \cdot 39$.

It seems worth while to try 344^2, which turns out to be 118336.

$$\begin{aligned} \therefore\ y &= \tfrac{1}{50}\{6 \pm 344\} \\ &= \tfrac{350}{50} \text{ or } -\tfrac{338}{50} \\ &= 7 \quad \text{ or } -\tfrac{169}{25}. \end{aligned}$$

[The factors thus are $(y-7)(25y+169)$.]

If $y=7$, $x=\frac{1}{4}(21-1)=5$.

If $y=-\frac{169}{25}$ or $-\frac{676}{100}$, $x=\frac{1}{4}(-20 \cdot 28 - 1)$
$= -5 \cdot 32$.

The solutions are

$$\left.\begin{aligned} x&=5 \\ y&=7 \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x&=-5 \cdot 32, \\ y&=-6 \cdot 76. \end{aligned}\right\}$$

Check. $x^2+y^2=25+49$ or $x^2+y^2=28 \cdot 30+45 \cdot 70$ from tables.
$=74$ $\qquad$ $=74 \cdot 0$

Examples 23. Solve the simultaneous equations :

1. $x^2=y+3$
$y=x+3$

2. $x-3y=7$
$x^2-4y^2+12y=9$.

3. $x^2=9y-2$
$2x=3y+1$.

4. $2x-3y=1$
$3xy+6x-42y=84$.

5. $x+y=0$
$3x^2+4xy=8y-33$.

6. $x^2-2y^2=7x-14$
$2x-y=7$.

7. $13y-4x=11$
$x^2-2xy-2y^2=22$.

8. $2y-9x=1$
$x^2+y^2+x+3y=250$.

Show that the following equations (Nos. 9, 10) have no solutions.

9. $x^2-2y^2=7x+14$
$2x-y=7$.

10. $x+y=20$
$xy=102$.

11. A rectangular field is 1 acre in area and its perimeter is 322 yards ; find its length and breadth.

12. The diagonal of a rectangle is 17 yards and its area is 120 square yards ; find the length of its sides.

13. The sum of the areas of two squares is 1898 sq. feet. The perimeters of the squares differ by 56 ft. Find the side of the larger one.

14. The square on the diagonal of a rectangle is greater than twice the area of the rectangle by 4 square yards. Show that the sides differ by 2 yards, and if the sides and the diagonal are each a whole number, find them.

Symmetric Equations

If $x+y$ and xy are given, all we know about x and y is that they are roots of a certain quadratic, taken in either order.

Thus if $x+y=7$ and $xy=12$, x and y are the roots of $z^2-7z+12=0$, that is x and y are 3 and 4 in either order.

This may be a convenient idea to use in such questions as those in Examples 24, but these can also be done by substituting in the usual way.

[Remember that $x^2+y^2=(x+y)^2-2xy$; $(x-y)^2=(x+y)^2-4xy$.]

Examples 24

1. The product of two numbers is 15 and the sum of their reciprocals is 8/15; find them.
2. The sum of two numbers is 20 and the sum of their squares 202; find them.
3. The sum of the squares of two numbers is 65 and the product of the numbers is 28; show that the numbers are the roots of
$$z^2 \pm 11z+28=0.$$
4. If the sum of the squares of two numbers is 34 and the sum of their reciprocals is 8/15, prove that the sum of the numbers is either 8 or $-17/4$, and in the latter case find the value of their product, and of their difference.
5. If $x+y=13$ and $x^3+y^3=559$, find xy by using the factors for the sum of two cubes. Hence find x and y.
6. Solve the equations $\left.\begin{aligned} x^2-xy+y^2=3 \\ x^2+xy+y^2=7 \end{aligned}\right\}$ in two ways:
   (i) by proving that $y/x=2$ or $\frac{1}{2}$ and completing the solution;
   (ii) by proving that $xy=2$ and $(x+y)^2=9$ and completing the solution.

Simultaneous Equations, both Quadratic

If each of a pair of simultaneous equations is a quadratic, elimination of one letter will normally give a fourth degree equation.

Thus given $y^2+2x=13$ and $x^2+xy+3x-5y=1$, we get
$$x=\tfrac{1}{2}(13-y^2); \quad \therefore \ \tfrac{1}{4}(13-y^2)^2+\tfrac{1}{2}y(13-y^2)+\tfrac{3}{2}(13-y^2)-5y=1,$$
which contains y^4 and is of the fourth degree.

Such equations cannot readily be solved "by quadratics".

There is, however, a special type which can readily be solved.

Simultaneous quadratics which are *unaltered if the signs of both* x *and* y *are changed* give a quadratic for the ratio $y:x$.

Put $y=mx$ and solve for m.

Example I. Solve $\left.\begin{aligned} x^2+2xy=12 \\ xy-2y^2=1 \end{aligned}\right\}$

Putting $y=mx$, these become

$$x^2+2mx^2 \ \ =12, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$
$$mx^2-2m^2x^2=1. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Eliminate the numbers on the R.H.S. by multiplying (ii) by 12 and subtracting

$$x^2 + 2mx^2 - 12mx^2 + 24m^2x^2 = 0.$$

We reject $x^2 = 0$, which does not give a solution of the original equations and get

$$1 - 10m + 24m^2 = 0 \text{ or } (1 - 6m)(1 - 4m) = 0.$$

If $m = \frac{1}{6}$, $x^2 + \frac{1}{3}x^2 = 12$, $x^2 = 9$, $x = 3$ or -3;

$\therefore\ y = \frac{1}{2}$ or $-\frac{1}{2}$.

If $m = \frac{1}{4}$, $x^2 + \frac{1}{2}x^2 = 12$, $x^2 = 8$, $x = 2\sqrt{2}$ or $-2\sqrt{2}$;

$\therefore\ y = \frac{1}{2}\sqrt{2}$ or $-\frac{1}{2}\sqrt{2}$.

There are four answers:

$$\left.\begin{matrix} x = 3, \\ y = \frac{1}{2}; \end{matrix}\right\} \quad \left.\begin{matrix} x = -3, \\ y = -\frac{1}{2}; \end{matrix}\right\} \quad \left.\begin{matrix} x = 2\sqrt{2}, \\ y = \frac{1}{2}\sqrt{2}; \end{matrix}\right\} \quad \left.\begin{matrix} x = -2\sqrt{2}, \\ y = -\frac{1}{2}\sqrt{2}. \end{matrix}\right\}$$

[It is not necessary to introduce a third letter; we can solve for $x : y$.]

Example II. Solve

$$x^2 + 2xy = 3, \quad \text{.........(i)}$$
$$y^2 - xy = 4. \quad \text{.........(ii)}$$

Multiply (i) by 4 and (ii) by 3 and subtract: we get

$$4x^2 + 11xy - 3y^2 = 0; \quad \text{.........(iii)}$$
$$\therefore\ (4x - y)(x + 3y) = 0;$$
$$\therefore\ y = 4x \text{ or } y = -\tfrac{1}{3}x.$$

If $y = 4x$, equation (i) reads $x^2 + 8x^2 = 3$

so that $3x = \pm\sqrt{3}$,

giving $x = \dfrac{\sqrt{3}}{3}$, $y = \dfrac{4\sqrt{3}}{3}$ or $x = -\dfrac{\sqrt{3}}{3}$, $y = -\dfrac{4\sqrt{3}}{3}$.

If $y = -\frac{1}{3}x$, equation (i), on multiplication by 3, reads

$3x^2 - 2x^2 = 9$, so that $x = \pm 3$,

giving $x = 3$, $y = -1$, or $x = -3$, $y = 1$.

Thus there are four solutions, but since there are only two values of $y : x$ the solutions are reached by solving quadratics.

It is interesting to note the graphical interpretation of the original equations and of the equation (iii) obtained from them.

The equation (i) is the equation of the curve (hyperbola) marked (i) in Fig. 7, while equation (ii) represents the curve (hyperbola) marked (ii).

Equation (iii) represents the pair of straight lines joining the intersections of the groups of (i) and (ii) to the origin.

Although the identification of the curves represented by equations (i) and (ii) is a problem of Co-ordinate Geometry rather than of

Algebra, yet it is important to realise the significance of the various steps in the solution.

The equations $y=4x$ and $x=-3y$ are the equations of two straight lines on which the intersections of the curves represented by equations (i) and (ii) lie.

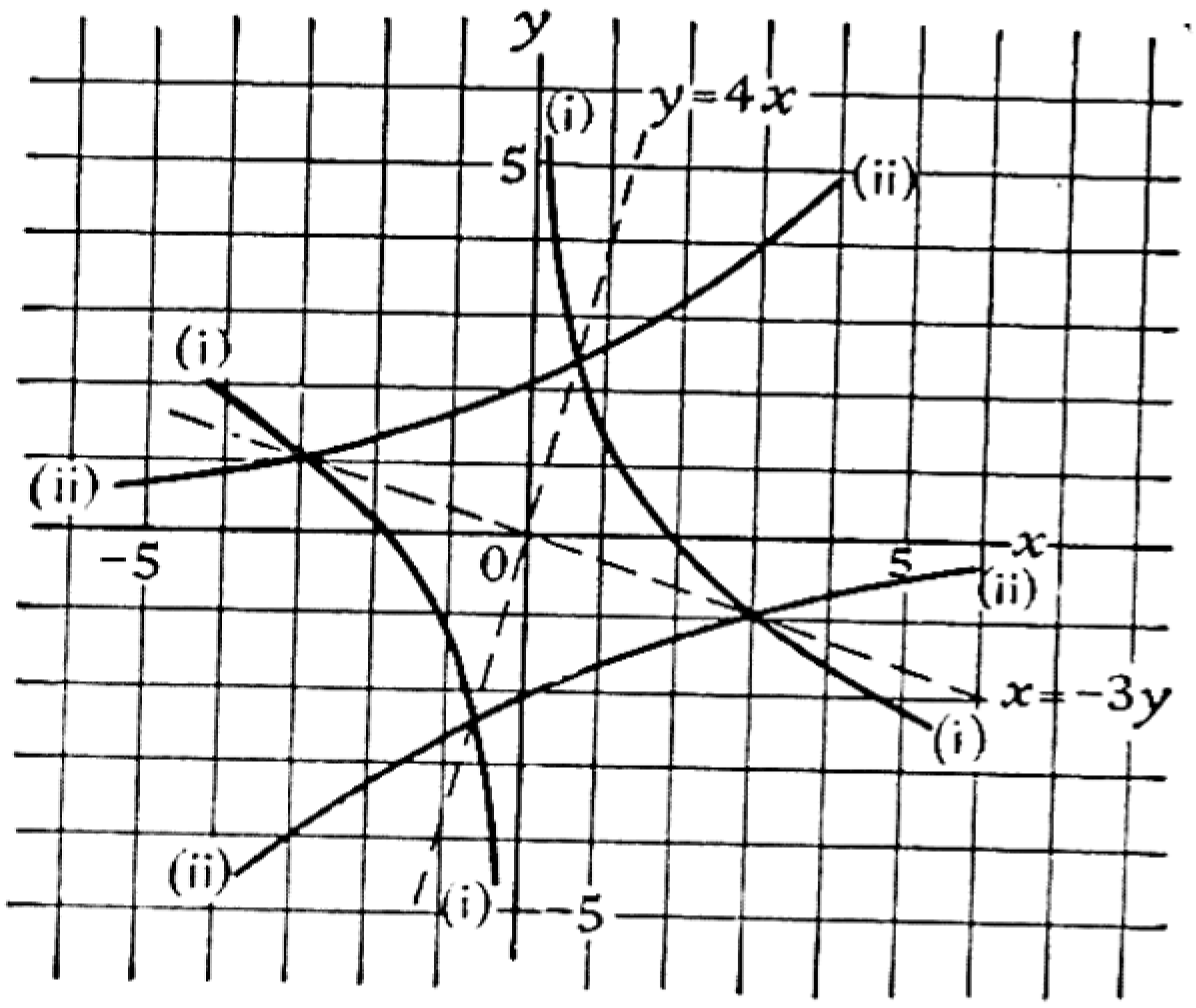

FIG. 7

Putting $y=4x$ in equation (i) we obtain the values of x at the points of intersection of $y=4x$ and $x^2+2xy=3$. To obtain the values of y we put $x=\pm\sqrt{3}/3$ in the equation $y=4x$. It would be harder to substitute in $x^2+2xy=3$, but this would give the same result. It would, however, be *wrong* to substitute in $y^2-xy=4$ as there are 4 points on this curve for which $x=\pm\sqrt{3}/3$, and only two of them are at the points of intersection with curve (i).

Examples 25. Solve the simultaneous equations: [Surds may be left in the answers to Nos. 1 to 5.]

1. $3x^2-2xy=8$, $y(x+y)=3$.

2. $x^2+y^2=25$, $3xy-2x^2=18$.

3. $6x^2-13xy+6y^2=0$, $5x^2-2y^2=2$.

4. $xy=9$, $x^2+4y^2=36$.

5. $x^2+y^2=16$, $x^2+4y^2=36$.

6. $4xy=9$, $x^2-y^2=20$.

7. $xy=1$, $x^2-y^2=4$.

8. $x^3-y^3=124$, $x-y=4$.

[In No. 7 prove that y/x, which must be positive, is $\sqrt{5}-2$; show that $x^2=\sqrt{5}+2$, $y^2=\sqrt{5}-2$, and give the answer in decimals.]

9. Show that if $S \equiv ax^2 + 2hxy + by^2$, then

$$aS = (ax + hy)^2 + (ab - h^2)y^2,$$

and that the coefficient of y^2 may be written $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$.

10. If $S_1 \equiv 7x^2 + 12xy + 5y^2$ and $S_2 \equiv 3x^2 - 14xy + 17y^2$ exhibit $7S_1$ and $3S_2$ as either the sum or difference of squares.

11. Show that the result of eliminating z between the equations

$$3x^2 + 2y^2 = 6z^2,$$
$$3x^2 - 14xy + 3y^2 = -20z^2,$$

can be written in the form $(39x - 21y)^2 + ky^2 = 0$ where $k = \begin{vmatrix} 39 & 21 \\ 21 & 29 \end{vmatrix}$.

Hence deduce that there are no values of $x : y : z$ satisfying the given equations.

12. Show that the simultaneous equations

$$x^2 + y^2 = r^2,$$
$$ax^2 + 2hxy + by^2 = 1,$$

only have real solutions if $r^4h^2 \geqslant (ar^2 - 1)(br^2 - 1)$.

Surds

Surds, of which $\sqrt{2}$ and $\sqrt{3}$ are the simplest examples, have been used in solving many of the previous quadratic equations. They are found to be a necessary extension of the numbers discussed in algebra, and are needed to express lengths such as those in Fig. 8.

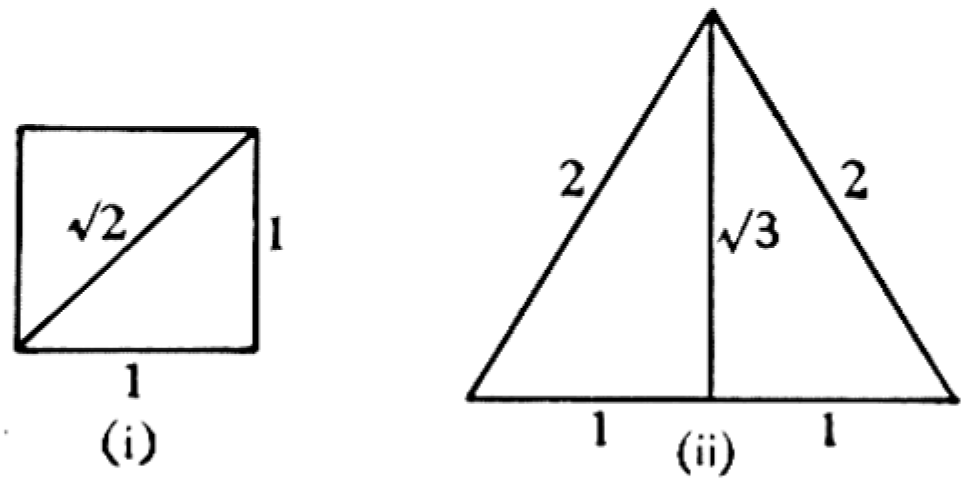

FIG. 8

Approximations to $\sqrt{2}$ can be found by showing that

$$1{\cdot}4^2 < 2,\ 1{\cdot}5^2 > 2;\quad 1{\cdot}41^2 < 2,\ 1{\cdot}42^2 > 2;\quad \text{and so on.}$$

This only involves the simple, though laborious, process of working out various squares, and we get $1{\cdot}41 < \sqrt{2} < 1{\cdot}42$.

Any who may have learnt the rule for square root can find approximations to 5 or 6 places of decimals without much trouble and, for instance, show that $1{\cdot}41421 < \sqrt{2} < 1{\cdot}41422$.

It can be proved that $\sqrt{2}$ is not a *rational* number, i.e. is not the ratio of two integers. (The proof dates back to Pythagoras.)

Suppose that $\sqrt{2}=\frac{a}{b}$ where a, b are integers without a common factor; this is not a limitation, for any common factor could be divided out. ....(H)

Then $a^2=2b^2$.

Hence a^2, being divisible by 2, must be even and so a is even. (The square of an odd number is odd.)

Since a is even, let $a=2c$;

$$\therefore\ 2b^2=4c^2 \text{ and so } b^2=2c^2.$$
$$\therefore\ b^2 \text{ is even, and hence } b \text{ is even.}$$

Thus a and b have a common factor 2, contrary to hypothesis (H).

$\therefore$ it is not possible for $\sqrt{2}$ to be of the form $\frac{a}{b}$.

$\therefore$ $\sqrt{2}$ is not rational and is said to be *irrational.*

It is to be understood that :—

(i) $\sqrt{2}$, and any other surd like it, obeys all the ordinary laws of algebra.

(ii) $(\sqrt{2})^2=2$.

(iii) $\sqrt{2}\,.\,\sqrt{3}=\sqrt{6}$, since the square of each side is 6.

Similarly for other surds.

Thus for $\sqrt[3]{2}$, which is read as "the cube root of 2", we have

$$(\sqrt[3]{2})^3=2 \text{ and } \sqrt[3]{2}\,.\,\sqrt[3]{3}=\sqrt[3]{6}\,; \quad \text{and again } (\sqrt[5]{3})^5=3.$$

Rationalising the Denominator

In finding decimal approximations to fractions containing surds, it is usually advantageous to make the denominator rational so as to avoid division by a decimal, as in the following:

(i) $\dfrac{5}{\sqrt{2}}=\dfrac{5\sqrt{2}}{2}\simeq\dfrac{5\times 1\cdot414}{2}=3\cdot535.$

(ii) $\dfrac{3}{\sqrt{2}-1}=\dfrac{3}{\sqrt{2}-1}\times\dfrac{\sqrt{2}+1}{\sqrt{2}+1}=\dfrac{3(\sqrt{2}+1)}{2-1}\simeq 7\cdot242.$

In (ii) the fraction is multiplied above and below by $\sqrt{2}+1$ which is called the *complementary* (*or conjugate*) *surd* of $\sqrt{2}-1$. It is convenient to give the name "surds" to such expressions as $\sqrt{2}+1$. When a quadratic equation with rational coefficients has roots which involve surds, the two roots are complementary surds.

(iii) $$\frac{3\sqrt{2}-4}{3-\sqrt{2}}=\frac{(3\sqrt{2}-4)(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}=\frac{(9-4)\sqrt{2}+6-12}{9-2}$$
$$=\frac{5\sqrt{2}-6}{7}\simeq\frac{1{\cdot}07}{7}\simeq 0{\cdot}15.$$

Examples 26

1. Express in the form $\sqrt{N}$: (i) $3\sqrt{2}$, (ii) $7\sqrt{3}$, (iii) $2\sqrt{15}$, (iv) $5\sqrt{2}\cdot\sqrt{3}$.
2. Express with the smallest possible number under the $\sqrt{\ }$ sign: (i) $\sqrt{27}$, (ii) $\sqrt{162}$, (iii) $\sqrt{84}$, (iv) $\sqrt{432}$.
3. Find the value, correct to 2 decimal places, of:

   (i) $\dfrac{8}{\sqrt{5}}$; (ii) $\dfrac{5}{4-\sqrt{6}}$; (iii) $\dfrac{\sqrt{3}+2}{\sqrt{3}-1}$; (iv) $\dfrac{2\sqrt{5}+\sqrt{3}}{3\sqrt{5}-2\sqrt{3}}$; (v) $\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{7}+\sqrt{2}}$;

   (vi) $\dfrac{6\sqrt{2}-\sqrt{6}}{2\sqrt{3}+3\sqrt{2}}$.
4. Prove that (i) $a\sqrt{b}=\sqrt{(a^2b)}$; (ii) $(\sqrt[3]{a})(\sqrt[3]{b})=\sqrt[3]{(ab)}$.
5. Give the proof that $\sqrt{3}$ is not rational.
6. Prove that $(\sqrt[3]{a})(\sqrt[4]{b})=\sqrt[12]{(a^4b^3)}$ and express $(\sqrt[3]{2})(\sqrt[4]{3})$ as the twelfth root of an integer.
7. Show that the ratios of the roots of $x^2-3x+1=0$ are the roots of $x^2-7x+1=0$, and that the ratios of the roots of $25x^2-20x+1=0$ are the roots of $x^2-14x+1=0$.
8. Show that if $\sqrt[3]{6}+\sqrt[3]{5}$ is multiplied by $\sqrt[3]{36}-\sqrt[3]{30}+\sqrt[3]{25}$ the result is rational, and state by what expression $\sqrt[3]{a}+\sqrt[3]{b}$ must be multiplied to give a rational result.

Forward reference

At this stage some teachers and students may like to discuss equations containing surds such as $\sqrt{2x+1}$; two worked solutions and equations to be solved will be found in Examples 27, Nos. 60–72.

Miscellaneous Examples 27

1. Solve the following equations, by factors:

   (i) $15x^2+25x=140$; (ii) $x+a=x(x+a)$;
   (iii) $3a^2x^2-10acx+7c^2=0$; (iv) $x^2-6px+9p^2=q^2$.
2. Solve the following equations. Surds may be left in the answers.

   (i) $x^2-5x-7=0$; (ii) $2x^2+7x+2=0$;
   (iii) $5x^2-16x+10=0$; (iv) $81x^2-234x+169=0$;
   (v) $9a^2x^2+36ax=1$; (vi) $136a^2x^2-103abx-14b^2=0$.
3. Form the quadratic equations whose roots are

   (i) $2\pm\sqrt{3}$; (ii) $\frac{1}{2}(-5\pm\sqrt{13})$.

4. Solve the equation $4x^2+4x-15=-a(4x^2-9)$ if
(i) $a=1$; (ii) $a=2$; (iii) $a=3$.
Why is one solution the same for all three cases?

5. Simplify $(4x^2+4x-15)\div(4x^2-9)$ and prove that the result is greater than 1 if x is positive and also if $-1\frac{1}{2}<x<0$.

6. Solve the general quadratic $ax^2+bx+c=0$ by (i) multiplying by $4a$, (ii) shifting $4ac$ to the R.H.S. and adding b^2 to each side.
[This is the Hindu method: the usual one is that of Omar Khayam, the Arabian astronomer-poet.]

7. Solve the equations: (i) $\frac{1}{3}(x+1)=8-\frac{1}{5}(x-1)$,
(ii) $\frac{1}{5}(2x^2-1)-\frac{1}{4}(3x+1)=9/10$.

8. A bill of £1 5s. 6d. is paid in half-crowns and shillings, there being 8 more shillings than half-crowns: how many are there of each?

9. Factorise: (i) x^2-5x+4; (ii) x^4+5x^2+4; (iii) x^4+3x^2+4.

10. Draw the graph of $y=x^2-3x-5$ and hence solve approximately the equation $x^2-3x-5=0$; solve this equation also by the formula.

11. Solve the equations: (i) $x+y=3$, $x^2+y^2=17$; (ii) $x+2y=2$, $4x^2+y^2=65$.

12. A man buys 60 tons of coal and 71 tons of coke for £705 12s. For £90 he gets 13 more tons of coke than coal. Find the price of each per ton.

13. (i) Form the equation whose roots are 4 and $-\frac{2}{3}$.
(ii) Solve the equations obtained from that in (i)
(*a*) by changing the sign of the term in x,
(*b*) by changing the sign of the absolute term.

14. The perimeter of a rectangular field is 4,400 yds. and its area is 240 acres. Find its length and breadth. [Work in chains.]

15. Write $3+6x-x^2$ in the form $a-(b-x)^2$.
For what value of x is it greatest and what is this greatest value?
Answer the corresponding questions for $3+6x+x^2$.

16. A body weighs 16·2 grams in air and 13·6 grams when immersed in water. Find its volume and specific gravity.

17. A number of two digits is equal to 7 times the sum of the digits; the difference between the squares of the digits is equal to 4/7 of the number. What is the number?
[Take $10x+y$ as the number.]

18. Find x to the nearest hundredth from
(i) $\dfrac{3\cdot2x+2\cdot1}{17}=\dfrac{3\cdot4x+5\cdot6}{14}$;
(ii) $2x^2+5x-11=0$.

19. A man goes a journey at an average speed of 24 m.p.h. If he had increased his average speed by 6 m.p.h. he would have taken 2 hours less; how far did he go?

20. (i) Form the equation whose roots are $-5\frac{1}{2}$, 3: also that whose roots are $2 \pm \sqrt{3}$.

(ii) Find $x : y$ if $2x^2 - 7xy + 6y^2 = 0$.

21. Factorise (i) $7xy - 4ab - 7ay + 4bx$; (ii) $4a^4 - 4a^2b^2 + 9b^4$.

22. Solve the equation $\begin{vmatrix} x & a \\ b & c \end{vmatrix} + \begin{vmatrix} a & b \\ c & x \end{vmatrix} = \begin{vmatrix} b & c \\ a & x \end{vmatrix}$.

23. A and B sell 100 sheep. If A sold his per head for twice as many shillings as B sold sheep and vice versa they would have between them received £480. How many sheep did each sell?

24. If 1 is added to the denominator of a fraction it is diminished by $\frac{3}{28}$. If 1 is added to the numerator of the same fraction it becomes unity. Find the fraction.

25. Prove the identity $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$.

[Taking any integers for m and n, this identity gives a rule for integral Pythagorean triangles; e.g. $m = 2$, $n = 1$ gives 3, 4, 5 for all sides.]

26. Prove that $\dfrac{a}{b} + 2$ and $\dfrac{2b}{a} + 2$ are such that the sum of their squares is an exact square, and that the same is true of $a(a + 2b)$ and $2b(b + a)$.

[Taking any integers for a and b this gives a rule for rational Pythagorean triangles. R. S. Williamson, *Mathematical Gazette*, Dec. 1953.]

27. If x and y are sides of a triangle and they contain a right angle and if $(x - 4)(y - 4) = 8$, prove that there are as many square units in the area as units in the perimeter of the triangle.

If x and y are integers, show that there are two solutions and find them.

[Ibid.]

28. If $2s = a + b + c$, work out $s(s - a)$ and $(s - b)(s - c)$ and show that if $a^2 = b^2 + c^2$, the results are equal, and that if a, b, c are also the sides of a triangle, each is the area of the triangle.

[The area of a triangle is always the square root of the product of these expressions.]

29. Draw two lines at right angles, $ABCD$ and $AEFG$, such that $AB = AE = a$, $BC = EF = b$, $CD = FG = c$; complete the square having these lines as adjacent sides and draw lines parallel to the sides through B, C, E, F. Use this diagram to evolve the expanded form of $(a + b + c)^2$.

30. If $P = a + b + c$ and $Q = a + b - c$ write out the expanded forms of P^2 and of Q^2 and find the value of $P^2 - Q^2$. Identify this with the product of $P + Q$ and $P - Q$. If $a + b = k^2c$, what is the square root of $P^2 - Q^2$?

31. The sum of the digits of a number less than 100 is 17, and the square of the ten's digit is less than the square of the unit's digit by 17 also. What is the number?

32. The difference between the squares of the two digits of a number is equal to their sum, which is itself one-fifth of the number. Find it.

33. Two men row a time race over a course of $1\frac{1}{4}$ miles and one wins by 10 seconds, his average speed being to that of the other as 45 : 44. Find the average speed of each in yards per second.

34. Find four consecutive numbers such that the product of the first and last is less by 19 than the number which has the two middle ones for its digits.

35. One side of a rectangle is 9 inches shorter than the diagonal and 9 inches longer than the other side. Find the area of the rectangle.

36. In Fig. 9, the curve labelled (i) is the graph of $y = ax^2$; what is the value of a?

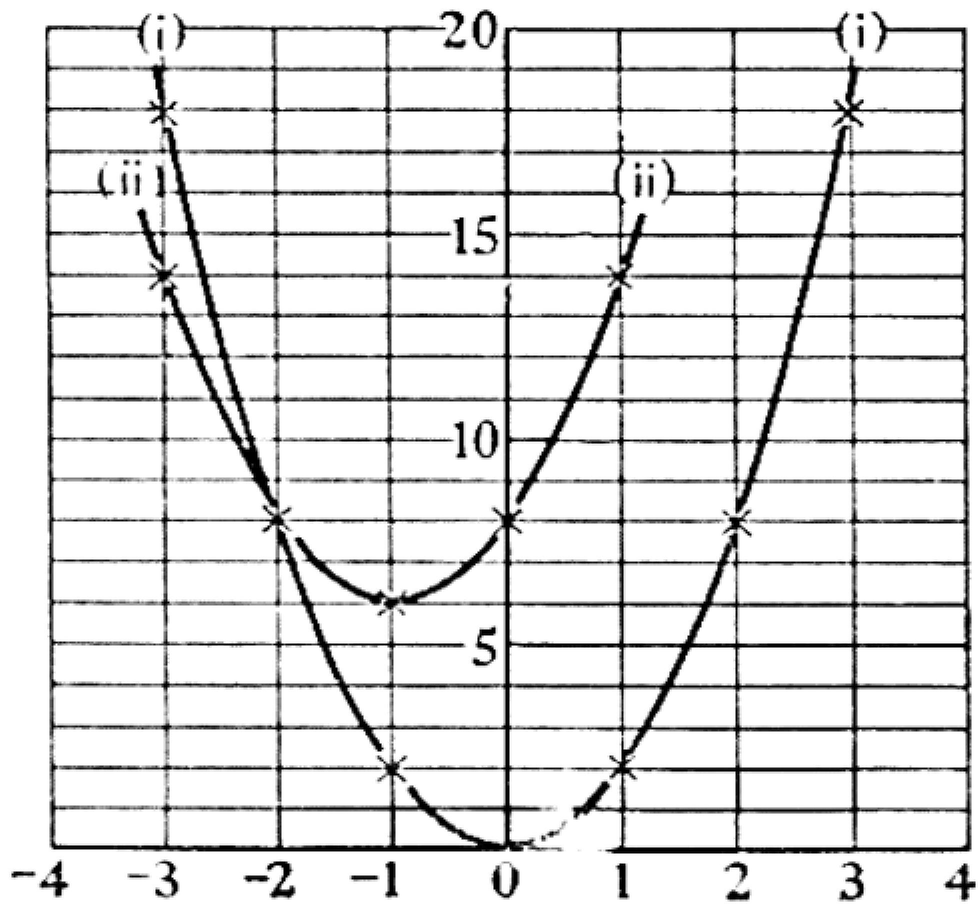

FIG. 9

37. In Fig. 9, the curve labelled (ii) is the graph of $y = a(x+b)^2 + c$; what are the values of a, b, c?

38. In Fig. 9 show that the equation $4x + 8 = 0$ should be satisfied where the curves meet (so that the apparent result $x = -2$ is exact, not approximate).

39. (i) Draw on the same diagram from $x = -5$ to $x = 5$ the graphs of $y = x^2$ and $y = 2x^2$, taking $\frac{1}{2}''$ as unit for x and $\frac{1}{5}''$ as unit for y.

(ii) In the diagram of (i) insert the graph of $y = 2x^2 + x + 1$.

What is the axis of symmetry of the new curve? What is the position of its vertex?

Check the results by writing $2x^2 + x + 1$ in the form $2(x+a)^2 + b$.

40. Find the values of a, b, c in order that the graph of $y = ax^2 + bx + c$ might pass through the points $(-1, 10)$, $(0, 5)$, $(1, 6)$; also find the position of its vertex.

41. Find the vertex of the curve $y = 3x - ax^2$.

Show that for varying a, all the curves touch the same line at the origin, and that their vertices lie on the line $y = \frac{3}{2} \cdot x$.

Solve the equations (Nos. 42 to 50):

42. $3x = 4y + 80$, $xy = 56y$.

43. $x - y = 1$, $xy = 56$.

44. $y = x + 2$, $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{9}{40}$.

45. $x^2 - y^2 = 16$, $2x - 3y = 1$.

46. $x^2 - y^2 = 9$, $2x^2 - 3xy + 2y^2 = 22$.

47. $xy = 10$, $y^2 - 9x^2 = 11$.

48. $x^2 + y^2 = 73$, $x^2 - y^2 = 55$.

49. $x^2 + y^2 = 73$, $11x^2 - 62y^2 = 146$.

50. $x^2 + y^2 = 73$, $(x + y)^2 - 16(x + y) + 55 = 0$.

51. If α and β are the roots of $x^2 + 2x - 5 = 0$, form the equations whose roots are (i) $\alpha^2 + \beta^2$, (ii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$, (iii) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$.

52. Find the value of c in order that the equation $(3x + c)^2 - 16x = 0$ may have equal roots.

53. Repeat Example 52 with the equation $x^2 + (mx + c)^2 = a^2$.

54. If one root of $x^2 + px + q = 0$ is five times the other root, prove that $5p^2 = 36q$.

55. If the roots of $x^2 + px + q = 0$ are α and β, prove that

(i) $(\alpha^2 + \beta^2) + p(\alpha + \beta) + 2q = 0$,

(ii) $(\alpha^3 + \beta^3) + p(\alpha^2 + \beta^2) + q(\alpha + \beta) = 0$,

(iii) $\alpha^n + \beta^n = -p(\alpha^{n-1} + \beta^{n-1}) - q(\alpha^{n-2} + \beta^{n-2})$.

56. Show that the square of $\sqrt{x} + \sqrt{y}$ will be $a + 2\sqrt{b}$ if

$$x + y = a \text{ and } xy = b.$$

Hence find the square roots of $8 + 2\sqrt{15}$ and $8 + 2\sqrt{7}$.

57. Find the square roots of (i) $11 + 2\sqrt{28}$, (ii) $15 - 2\sqrt{54}$, (iii) $10 - \sqrt{84}$.

58. If $x = \frac{1}{4}(\sqrt{5} - 1)$ prove that $1 - 2x^2 = \frac{1}{4}(\sqrt{5} + 1)$.

59. If $x = 3 - \sqrt{3}$ prove that $x^2 + \dfrac{36}{x^2} = 24$.

[In solving equations containing surds it is necessary to remove the surd by squaring, an operation which may produce other solutions; consequently it is *essential to test by substitution* which of the solutions satisfy the original equation. It is understood that the sign $\sqrt{}$ means the *positive* square root.]

Solve equations 60–71. (See note at end of p. 49.)

60. (i) $x=\sqrt{5x-6}$; (ii) $\sqrt{2x+6}-\sqrt{x+4}=1$.

[(i) Solution. Squaring both sides gives $x^2=5x-6$.

$$\therefore\ x^2-5x+6=0;\quad \therefore\ (x-3)(x-2)=0.$$

$$\therefore\ x=3 \text{ or } x=2.$$

If $x=3$, L.H.S. $=3$, R.H.S. $=\sqrt{15-6}=\sqrt{9}=3$; $\therefore$ $x=3$ satisfies.
If $x=2$, L.H.S. $=2$, R.H.S. $=\sqrt{10-6}=\sqrt{4}=2$; $\therefore$ $x=2$ satisfies.

$$\therefore \text{ either } x=3 \text{ or } x=2.$$

(ii) Solution. $\sqrt{2x+6}=1+\sqrt{x+4}$; $\therefore$ by squaring both sides

$$2x+6=1+2\sqrt{x+4}+x+4.$$

$\therefore$ $x+1=2\sqrt{x+4}$ which when squared gives

$$x^2+2x+1=4(x+4).$$

[$\sqrt{2x+6}+\sqrt{x+4}=1$ would also have yielded this equation.]

$\therefore$ $x^2-2x-15=0$, giving $x=5$ or $x=-3$.

If $x=5$, L.H.S. $=\sqrt{16}-\sqrt{9}=4-3=1=$ R.H.S. $\therefore$ $x=5$ satisfies.

If $x=-3$, L.H.S. $=\sqrt{0}-\sqrt{1}=-1$; R.H.S. $=1$. $\therefore$ $x=-3$ does not satisfy.

$$\therefore\ x=5 \text{ is the solution.]}$$

61. $\sqrt{x^2-7}=7-x$.

62. $2x-\sqrt{2x-1}=x+2$.

63. $\sqrt{3x+10}-\sqrt{2x-1}=2$.

64. $\sqrt{5x-1}+\sqrt{7x-6}=\sqrt{20x-25}$.

65. $\sqrt{3x+1}-\sqrt{2x}=9$.

66. $\sqrt{2x}-\sqrt{x+1}=1$.

67. $\sqrt{2x-1}+\sqrt{3x+1}=7$.

68. $\sqrt{3x+1}-\sqrt{2x-1}=7$.

69. $\sqrt{4x+4}-\sqrt{x+1}=3$.

70. $\sqrt{5x+9}-\sqrt{3x+1}=\sqrt{2x-12}$.

71. $2(a+b-2x)=5\sqrt{(a-x)(b-x)}$.

72. Show that the equation $\sqrt{x+9}=\sqrt{5x+1}-2$ has only one root, which is the same as that of $\sqrt{x+9}=\sqrt{x-3}+2$.

Also show that the equation $\sqrt{5x+1}-2=\sqrt{x-3}+2$ has a different root.

Reciprocal Equations

[Equations such as Nos. 73 to 76 below, which are unaltered if x is replaced by $\frac{1}{x}$ are called *reciprocal* equations. Try to solve for $x+\frac{1}{x}$].

Solve the equations:

73. $x^4-x^3-4x^2-x+1=0$.

[Solution. Divide by x^2 and arrange as $x^2+\frac{1}{x^2}-\left(x+\frac{1}{x}\right)-4=0$.

If $x+\frac{1}{x}=y$, this is $y^2-2-y-4=0$ or $y^2-y-6=0$,

$$\text{giving } y=3 \text{ or } -2;$$

$$x \text{ is given by } x+\frac{1}{x}=3 \text{ or } -2,$$

$$\text{i.e. } x^2-3x+1=0 \quad \text{or} \quad x^2+2x+1=0.$$

The answers are $x=\frac{1}{2}(3\pm\sqrt{5})$ or -1 repeated.]

74. $x^4+4x^3+6x^2+4x+1=0$.

75. $4x^4-8x^3+3x^2-8x+4=0$.

76. $x^4-3x^3+4x^2-3x+1=0$.

77. By putting $x-\frac{1}{x}=y$ solve the equation

$$x^4-2x^3-5x^2+2x+1=0.$$

Symmetrical Equations

78. By putting $x+y=p$ and $xy=q$ show that the equations

$$x^2+y^2=4(x+y)+9, \quad x^2-5xy+y^2=x+y,$$

give $p=7$, $q=6$ or $p=-9/5$, $q=18/25$.
Hence find x and y.

79. Solve, as in No. 72, 164 Pb.

$$x^2-xy+y^2=19, \quad xy-3=x+y.$$

80. Solve $(x+y)^2+x^2y^2=37$; $\quad x+y=xy+5$.

81. Find x, y, z given

$$bcx=cay=abz \text{ and } ayz+bzx+cxy=1.$$

State the condition a, b, c must satisfy for your solutions to exist.

82. Given $pq\neq 0$ find x, y, z if $x(y+z)=y(z+x)=p$ and $z(x+y)=q$.

Some examination examples:

83. Show that for all (real) values of k, the equation

$$4x^2-4(k+3)x+5k+8=0$$

has (real) roots. (L.)

84. Find the values of a for which the expression $(2a+3)x^2-6x+4-a$ is a perfect square. (L.)

85. If each of the equations

$$x^2+(2a-b)x+ab=0 \quad \text{and} \quad x^2+(4a-b)x+k=0$$

has equal roots, express k in terms of a. (L.)

86. Determine the values of k for which the expression

$$k(x^2+2x)-4x+3k$$

is greater than 2 for all (real) values of x. (L.)

87. If α, β are the roots of the equation $ax^2+bx+c=0$ construct the quadratic equation whose roots are $2\alpha+\frac{1}{\beta}$, $2\beta+\frac{1}{\alpha}$; and prove that when c is equal to $-a$ this quadratic is the same as the original quadratic. (N.)

88. If the sum of the squares of the roots of $x^2+bx+c=0$ is p^2 and the sum of the reciprocals of the root is $1/q$, prove that $b^2+2bq-p^2=0$. (L.)

89. If the equations $x^2+px+r=0$ and $x^2+rx+p=0$, where $p \neq r$, have a common root and the other root of the first equation is k times the other root of the second, prove that $p=-\frac{1}{1+k}$. (L.)

90. Solve the equation $\frac{(1-x)^2}{2-x^2}=\frac{(1-a)^2}{2-a^2}$.

Prove that if x has a value not equal to a, then a cannot be equal to 1 or 2. (N.)

91. If α, β are the roots of the equation $x^2-px+q=0$ prove, without solving the equation, that $\sqrt{\left(\frac{\alpha}{\beta}\right)}+\sqrt{\left(\frac{\beta}{\alpha}\right)}=\frac{p}{\sqrt{q}}$. (L.)

Obtain the equation whose roots are $\alpha+3$, $\beta+3$. (L.)

92. If α, β are the roots of the equation $ax^2+bx+c=0$, find the condition that $\alpha^3+\beta^3=1$.

93. Prove that the expression $3x^2+8x+6$ is always positive and that its least value is $\frac{2}{3}$.

94. Find the equation whose roots are $\alpha-\frac{2}{\beta}$, $\beta-\frac{2}{\alpha}$, where α, β are the roots of $x^2-4x+8=0$. (N.)

95. Show that there are two values of k for which the equation

$$\frac{a}{x+c}+\frac{k}{x}+\frac{b}{x-c}=0$$

has equal roots in x, that the product of these values of k is $\frac{1}{4}(a-b)^2$ and that the product of the two corresponding values of x is c^2. (O. & C.)

96. Prove that the roots of the equation $(x-a)(x-b)-h^2=0$ are real and different.

Show also that the squares of the roots of this equation are the roots of the equation $(y-a^2-h^2)(y-b^2-h^2)=h^2(a+b)^2$. (L.)

CHAPTER III

VARIATION. GRAPHS. NUMERICAL EQUATIONS

Relations between Two Variables (Variation)

To say that "y is a function of x" means that x and y are both variables, and that the value of y is determined by the value of x.

Some phrases used for special types of relation will now be given.

(i) If $y = ax$ where a is a constant, two phrases are in common use to describe the relation: *y varies as x* * and *y is proportional to x*.

The symbolic notation $y \propto x$ is also often used.

For instance, if a series of sticks are placed upright, then the lengths of their shadows cast by the sun at the same moment will be proportional to the lengths of the sticks.

Again *y varies as x^2* or $y \propto x^2$ implies that $y = kx^2$ where k is a constant; here y is proportional to the square of x.

As examples:

the circumference of a circle varies as its diameter ($C = \pi d$),

the area of a circle varies as the square of its diameter ($A = \frac{1}{4}\pi d^2$).

(ii) If $y = ax + b$ where a and b are constants, we say there is a *linear relation* between x and y since the graph showing the relationship is a straight line.

In this case, if two values of x are taken, the *change* in the value of y is *proportional to the change* in the value of x.

For if x_1, y_1 and x_2, y_2 are pairs of values,

$$\left.\begin{aligned} y_1 &= ax_1 + b \\ y_2 &= ax_2 + b \end{aligned}\right\} \text{ so that } y_2 - y_1 = a(x_2 - x_1).$$

A relationship of this kind exists between corresponding Fahrenheit and Centigrade temperatures: $F = \frac{9}{5}C + 32$.

(iii) If $y = \dfrac{a}{x}$, y is said to *vary inversely as x*, or to *vary as $\dfrac{1}{x}$*.

Sometimes the phrase y varies *directly* as x is used for y varies as x in order to make a more vivid contrast with the case of inverse variation. Thus, for a journey, the time taken varies directly as the

* This is the shorter phrase, but it is liable to be confused with "y varies as x varies", which is true if y is any function of x.

distance for a given average speed, but varies inversely as the average speed for a given distance.

(iv) If more than one independent variable is concerned, similar phrases are used:

If $y=axz$, y varies as x and also varies as z, or y may be said to vary *jointly* as x and z.

If $y=\dfrac{ax}{z^2}$, y is said to vary *directly* as x and *inversely* as the square of z.

In Fig. 10, which is a rectangle of fixed length l and with variable width having semicircles described on its ends as diameters,

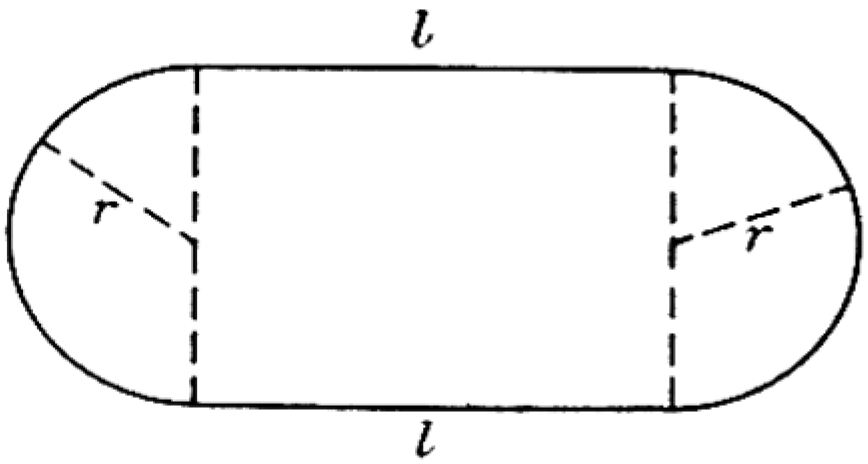

FIG. 10

the perimeter $(p)=2l+2\pi r$,

i.e. p is the sum of a constant and a term which varies as r;

the area $(A)=2lr+\pi r^2$,

i.e. A is the sum of two terms, one of which varies as r and the other varies as r^2.

Applications

The phrases just introduced are especially useful in mensuration and in physics.

It may be known from general considerations that one quantity must vary as another, or as its square, directly or inversely.

The relation between them can then be written down in a form involving an unknown constant, and this constant determined by experiment.

For example, the weight (w gm.) of a square of cardboard cut from a uniform sheet will vary as the square of the side (x cm.) of the square; so that $w=kx^2$ where k is a constant.

Now if a square of side 2 cm. is found to weigh 9 gm., it follows that $9=k\cdot 2^2$ or $k=9/4$, so that $w=9x^2/4$. This will be the general relation between the weight and the side of any square cut from this sheet of cardboard.

When two constants are involved, as in the linear relation $y = ax + b$, two pairs of corresponding values must be known before a and b can be determined (two points fix a line).

In examples, if we are given not only the general law but also a sufficient number of special results, we can

(i) translate the general law into algebraic form,
(ii) use the special results to determine the constants.

Example I. The surface of a sphere is known to vary as the square of the diameter.

Being given that a sphere of diameter 5 inches has a surface 78·54 sq. inches find the general formula and calculate the surface of a sphere of diameter 9 inches.

Solution. Surface varies as diameter squared or $S \propto d^2$;

$$\therefore\ S = kd^2 \text{ where } k \text{ is a constant.}$$

Substituting the given values, $78{\cdot}54 = k \,.\, 25$;

$$\begin{aligned}\therefore\ k &= 78{\cdot}54 \div 25\\ &= 78{\cdot}54 \times 4 \div 100\\ &= 3{\cdot}1416.\end{aligned}$$

$\therefore$ the general formula is $S = 3{\cdot}1416d^2$.

If $d = 9$, $S = 3{\cdot}1416 \times 81 = 254{\cdot}47$.

Note. If the general formula were *not* required the second answer could be found from $S = k \,.\, 9^2$ and $78{\cdot}54 = k \,.\, 5^2$ which lead to

$$\frac{S}{78{\cdot}54} = \frac{k \,.\, 9^2}{k \,.\, 5^2} = \frac{81}{25},$$

$$\text{i.e. } S = 78{\cdot}54 \times \frac{81}{25} = 254{\cdot}47.$$

Example II. The number n of vibrations per second of a stretched string varies as the square root of its tension T and inversely as its length l. Give a formula for this.

For a special string the number of vibrations is 181. What is the rate of vibration if the length is halved and the tension reduced by one-third?

Solution. The formula is $n = a\sqrt{T}/l$ where $181 = a\sqrt{T}/l$ for the given string.

If x is the required number, $x = \dfrac{a\sqrt{2T/3}}{\frac{1}{2}l}$.

Hence by division $\dfrac{x}{181} = \dfrac{\sqrt{\frac{2}{3}}}{\frac{1}{2}}$.

$$\begin{aligned}\therefore\ x &= 181 \times 2 \times \sqrt{\tfrac{2}{3}}\\ &= 295{\cdot}6 \text{ approx.}\end{aligned}$$

Examples 28

1. Write down formulae (containing undetermined constants) corresponding to the following statements :

(i) The volume (V) of a sphere varies as the cube of its diameter (D).

(ii) The attraction (A) between two bodies varies as the product of their masses (M, M') and inversely as the square of the distance (D) between them.

(iii) The light (L) received on a given area is proportional to the area (A) and inversely proportional to the square of the distance (D) from the source of the light.

(iv) The resistance (R) to the motion of a train is the sum of a constant and a term proportional to the square of its velocity (V).

(v) The electrical resistance (R) of a wire of given material varies directly as its length (l) and inversely as the square of its diameter (d).

(vi) The surface (S) of a cylinder varies as the product of the radius (r) of the base and the sum of this radius and the height (h).

2. Determine the constants in the formulae given as answers to parts (i), (iv), (vi) of No. 1 being the following :
for (i) that the volume of a sphere of radius 3 inches is 113·1 cubic inches,
for (iv) that the resistance at 20 miles per hour is 1,300 lb., while the resistance at 60 miles per hour is 2,100 lb.,
for (vi) that the surface of a cylinder whose height is 3 inches and the radius of whose base is 4 inches is 175·9 square inches.

3. If $y \propto x^2$ and y is 6 when x is 1·2 find the formula connecting y with x and use it to find the value of y when x is 3·6. Find the last result also directly from the general statement without using the value of the constant in the formula.

4. If y varies inversely as x^3 and $y=8$ when $x=2$, for what values of x is y less than unity?

5. If z varies jointly as x and $\sqrt{y}$ what is the effect on z of (i) doubling x and halving y? (ii) doubling y and halving x? [The data mean $y \propto x\sqrt{y}$.]

6. State the types of variation by which the following relations may be expressed :

(i) the relation between the volume of a cone and the height (the radius of the base being given),

(ii) the relation between the volume of a cone and the radius of its base (the height being given),

(iii) the relation between the time taken on a journey and the average speed.

7. Are the following values consistent with the "law of the inverse square", i.e. with the statement that y varies inversely as x^2?

x	1	2	3	4	10
y	720	180	80	42	7·2

If not, alter one or more of the values of y so that they may be consistent with that law.

8. What law of variation is suggested by the table below?

x	1	2	3
y	1·1	4·4	9·9

State the law with y as the subject of the sentence, and also with x as the subject of the sentence.

Find x if y is 22.

9. The electrical resistance of a wire of given material varies directly as its length and inversely as the square of its radius.

Find (i) the length of wire of radius 0·25 mm. having the same resistance as 100 metres of wire of the same material and of radius 0·5 mm.

(ii) the radius of a wire having the same length and twice the resistance of wire of the same material of radius 0·2 mm.

10. A number of circular discs of different diameters are cut from a thin sheet of metal of uniform thickness. What kind of relation exists between the weights (w oz.) and the diameters (d in.)?

If the disc of diameter 2 inches weighs 12 oz., find

(a) the weight of a disc of diameter (i) 3 in., (ii) 3·5 in.;

(b) the diameter of a disc weighing (i) 75 oz., (ii) 119·07 oz.

11. Coils of copper wire each 100 metres long and of circular section have different diameters. The weight of the coil of wire with diameter 2 mm. is 2·83 kg.

(a) What would be the weight of the coil of wire having diameter (i) 3 mm.? (ii) 1 mm.? (Give answer to 2 decimal places.)

(b) What would be the diameter of the wire in a coil weighing 5·66 kg.?

12. A piece of elastic is suspended vertically from one end and various weights are attached to the other end, extending the elastic. The length (l cm.) and the weight (w gm.) are connected by a relation of the form $l = a + bw$, where a and b are constants. When $w = 3$, $l = 13·2$, and when $w = 9$, $l = 15·6$. Find a and b and the value of w which will make $l = 16·8$.

13. The weight of water a cubical tank will hold varies as the cube of its length, while the weight of zinc required to line the tank varies as the square of its length. State the relation between the weight of water (W lb.) and the weight of the zinc (Z lb.).

14. The value of a diamond varies as the square of its weight. What percentage of its value is lost if a diamond of weight $(b+c)$ units is cut into two parts of weight b and c units respectively?

15. The force of attraction (F) which the sun exerts on one of its planets varies directly as the mass (m) of the planet and inversely as the square of the planet's distance (d) from the sun.

Alternatively the force (F) varies directly as the distance (d) and inversely as the square of the time (t) which the satellite takes to go round the sun. Show that $m \propto d^3/t^2$.

16. A number of circular cylinders of differing radius (r cm.) and differing length (l cm.) are made of the same metal. If the weight of a cylinder is w gm., what kind of relation exists between w, r, and l?

One of the cylinders has radius 1·5 cm., length 4 cm., and weight 81 gm. Find :

(i) the weight of a cylinder of radius 1 cm. and length 6 cm. ;

(ii) the length of a cylinder of radius 2 cm. and weight 108 gm. ;

(iii) the radius of a cylinder of length 5 cm. and weight 115·2 gm.

Laws from Experimental Data

In an experiment designed to discover the relation (if any) between two variables x and y, the following readings were obtained :

x	9	11	14	17	19	22	24	28	31
y	23	29	36	43	48	56	63	71	79

The points plotted in Fig. 11 suggest that a *linear* relation exists between the variables x and y since a line can be drawn such that, while it does not pass through all the points, yet making allowance for experimental errors, it can be said to fit the readings.

In this case, the line passes through the origin, so $y = kx$ for some value of k. Using the point P on the line (P given by $x = 19$, $y = 48$), $48 = k \cdot 19$ and so $k = 48/19 = 2{\cdot}53$ approx. Therefore the relation between the variables is approximately $\mathbf{y = 2{\cdot}53x}$.

In Fig. 12 showing points fixed by pairs of values of W and x, the line which the points suggest does not pass through the origin. Here, the increase in x is proportional to the increase in W, and so

using the point L (0, 1·2), $x-1{\cdot}2=k\,.\,W$ where k is some constant number.

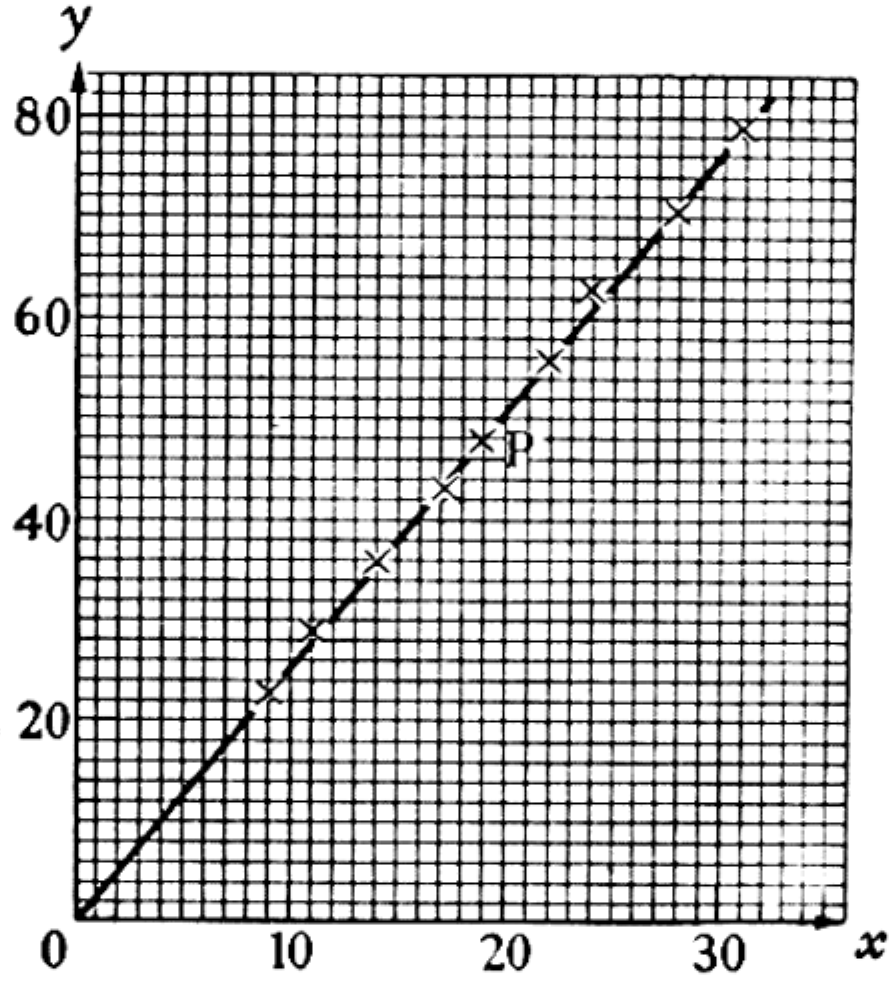

FIG. 11

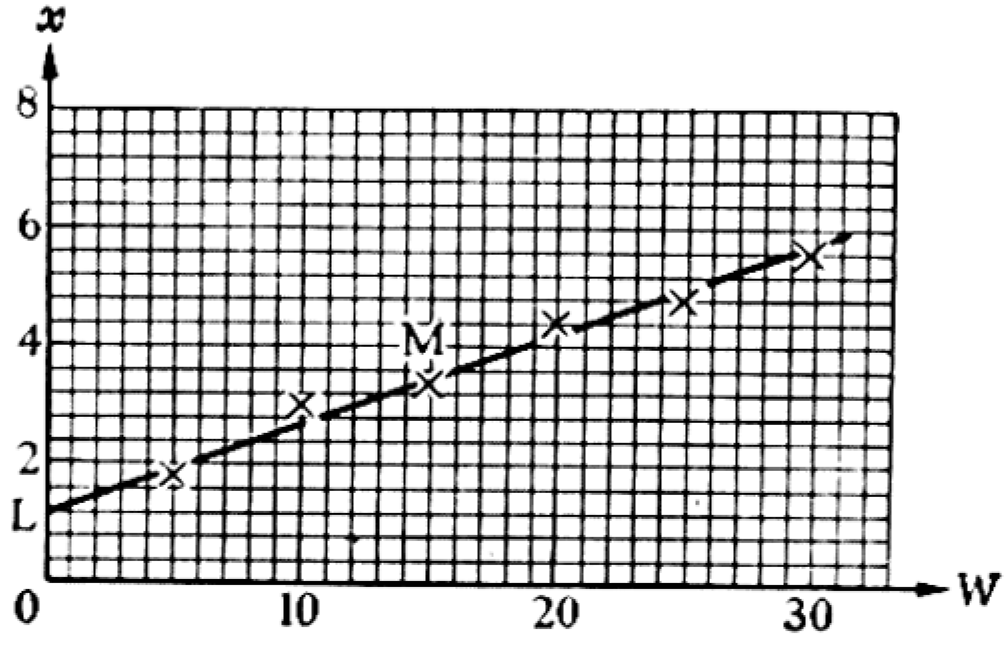

FIG. 12

Now using the point M (15, 3·4)

$$3{\cdot}4-1{\cdot}2=k\,.\,15, \text{ whence } k=2{\cdot}2/15=0{\cdot}15 \text{ approx.}$$

Therefore the relation between the variables is $\mathbf{x-1{\cdot}2=0{\cdot}15}\ W$.

Sometimes a straight line graph will be given by plotting functions of the variables against each other, as in the following instances:

Law $xy=k$, then $y=k\left(\frac{1}{x}\right)$ and a straight line will be given by plotting y against $\frac{1}{x}$.

Law $x^2y=k$, then $y=k\left(\frac{1}{x^2}\right)$ and a straight line will be given by plotting y against $\frac{1}{x^2}$.

* Law $y=kx^n$, then $\log y=\log k+n\log x$ and a straight line will be given by plotting $\log y$ against $\log x$. Denoting $\log y$ by Y and $\log x$ by X the relation becomes $Y=nX+$ a constant number, i.e. the relation is linear in X and Y.

* Law $y=ka^x$, then $\log y=\log k+x\log a$, and a straight line will be given by plotting $\log y$ against x.

Examples 29

1. State how straight-line graphs may be obtained from sets of figures obeying the following laws:

(i) $T=\cdot4\sqrt{l}$; (ii) $v^2=12r$; (iii) $rn^2=2\cdot7$;

(iv) $S=\frac{15}{(3+x)}$. (v) $y=5-x^2$; (vi) $y^2=x^3$.

[Solution (i). Plot T against $\sqrt{l}$.
Alternatively, since $\log T=\log(\cdot4)+\frac{1}{2}\log l$, plot $\log T$ against $\log l$.]

2. State the law independent of logarithms in each of the following cases:

(i) $\log y=\cdot301+2\log x$; (ii) $\log y=2-\log x$;
(iii) $\log y=x$; (iv) $3\log y=2\log x$;
(v) $\log y=1-x\log 2$; (vi) $2\log y=4-3\log x$.

3. For the relation $xy=40$ make a table of values of x and y for $x=2$, 4, 5, 8, 10 and plot y against x.

Now make a table of values of $\frac{1}{x}$ and y; taking 5 small squares to represent $\cdot1$ unit along the $\left(\frac{1}{x}\right)$-axis and 5 units along the y-axis, plot y against $\frac{1}{x}$ to give points in a straight line.

Also make a table of values of $\log x$ and $\log y$ for the same values of x; taking 5 small squares to represent $\cdot1$ along each of the axes, plot $\log y$ against $\log x$ to give points in a straight line.

* It is here assumed that the reader is familiar with the use of logarithmic tables.

4. The variables x and y have the pairs of values given in the following table :

$x=$	1	2	3	4	5	6
$y=$	32	16	8	4	2	1

First plot y and x; next plot $\log y$ against x and obtain a straight line graph. What relation exists between x and y?

5. The variables t and s have the pairs of values given in the following table :

$t=$	1	2	3	4	5	6
$s=$	2	7	16	29	45	65

Plot s against t^2 to give an approximate straight-line graph, and find a law which the pairs of numbers obey approximately.

6. A metre rule was suspended symmetrically by two vertical threads of the same length from two points at the same level. The rod was then twisted through a small angle and released so as to perform small oscillations about the vertical through its mid-point. The distance apart (d cm.) of the points of attachment and the time of oscillation (T sec.) were recorded to give the following table :

d cm.	97	80	70	60	50	40	30	20	10
T sec.	·816	·990	1·124	1·334	1·600	2·02	2·56	3·90	7·69

Show that T plotted against $\frac{1}{d}$ gives a straight-line graph.

7. A metal ball was rolled down an inclined plane and the time taken for it to travel varying distances were measured by means of a water clock. The following table resulted :

Distance rolled (s cm.)	80	120	160	200	240	280	310
Time taken (v cc.)	45	55	65	74	82	89	$93\frac{1}{2}$

First plot s against v^2 and then $\log v$ against $\log s$; deduce from each that there is a relation of the form $s=kv^2$.

8. A long flat spring was clamped at varying distances from the free end and the time for horizontal vibrations recorded to give the following table:

Length vibrating (l cm.)	35	40	45	50	55	60	65	70
Time of a vibration (T sec.)	·35	·44	·57	·67	·85	·99	1·12	1·32

Test whether $T \propto l^n$ by plotting log T against log l.

9. A deflection magnetometer experiment was designed to confirm the inverse square law. Use the following table to show tan $\theta \propto \frac{1}{d^2}$.

d cm.	13	15	17	19	21	23	25
θ degrees	70	64	58	52	45	40	35

10. Show that the following table confirms that when the temperature of a quantity of gas is kept constant its pressure varies inversely as its volume.

Pressure (cm. of Hg.)	90·4	93·8	95·2	99·5	102
Volume cc.	23·8	22·8	22·4	21·7	21

Pressure (cm. of Hg.)	103·6	105·5	107·2	109·8	112·3
Volume cc.	20·7	20·4	20·1	19·6	19·2

11. A thin strip of metal is held horizontally and then one end is clamped. Weights are hung from the free end and the consequent depression measured. Determine the law connecting depression with weight from the following table:

W (gm.)	5	10	15	20	25	30
D (cm.)	·47	·74	1	1·26	1·52	1·80

12. The electrical resistance (R ohms) of an iron wire was measured at varying temperature ($T°$ C.). Find the linear relation which exists between R and T from the table:

T	37·5	44·5	50	52	57	61	70	79
R	10·6	10·8	11	11·1	11·2	11·4	11·6	11·9

13. Variables x and y are related by a law of the form $y = kx^n$. Approximate values of y for various values of x are given by the table:

x	3	$4\frac{1}{4}$	$5\frac{1}{2}$	8	10	11	12
y	22	26	30	36	40	42	44

From the graph of $\log y$ against $\log x$, deduce the values of the constants k and n.

14. The speed of an electric train during the first two minutes after start is given by:

Time in sec. (t)	0	10	25	45	70	95	120
Speed in ml./hr. (v)	0	15	25	37	47	54·5	57·7

If these figures are represented approximately by the formula $(a-t)^2 = b(c-v)$, find the best values of a, b, c.

15. A weight was hung by a light string of length l cm. from a fixed point and allowed to perform small swings about the vertical. The time for 20 complete swings was recorded and so the time (T sec.) for 1 swing calculated. Find whether the following table supports the theory that $T^2 \propto l$.

l	158	110	91	$55\frac{1}{2}$	34
T	2·54	2·14	1·95	1·54	1·24

Cubic Functions

Graphs

The general cubic function of x is given by

$$y = ax^3 + bx^2 + cx + d$$

in which, for the present, a will be assumed to be positive.

The graph of this function has the following properties:

(i) If x is large and positive, y is large and positive.
If x is large and negative, y is large and negative.

(ii) Hence the graph must cross the x-axis at least once; if this is where $x = p$, then $x - p$ is a factor of y.

(iii) The other factor of y must be a quadratic, which may or may not have real factors;

$$\therefore\ y = a(x-p)(x-q)(x-r),$$

or else $$y = a(x-p)\{(x-s)^2 + t^2\}.$$

In the former case if $p<q<r$ the graph crosses the axis going upwards where $x=p$, downwards where $x=q$ and upwards again where $x=r$.

The graph is therefore of the type (i) shown in Fig. 13. It must have a maximum between $x=p$ and $x=q$ and a mimimum between $x=q$ and $x=r$.

(iv) The gradient function (or derivative) of y is quadratic given by $\dfrac{dy}{dx}=3ax^2+2bx+c$.

This may or may not give real turning points.

In the functional notation, if $f(x)=ax^3+bx^2+cx+d$

$$\text{then } f'(x)=3ax^2+2bx+c.$$

(v) There will in all cases be a real point of *inflexion* (see Fig. 13) given by $\dfrac{d^2y}{dx^2}=6ax+2b=0$; i.e. where $x=-b/3a$.

If a is negative the curve will run from high up on the left to low down on the right, instead of the other way round.

Since the curve can be written $\dfrac{y}{a}=x^3+\dfrac{b}{a}x^2+\dfrac{c}{a}x+\dfrac{d}{a}$ there is no loss of generality other than a change of scale if the leading term for y is taken to be x^3 instead of ax^3.

Notation

The property (i) is often expressed thus :

As $x\to\infty$ then $y\to\infty$; as $x\to-\infty$ then $y\to-\infty$.

The first of these sentences is read " *As x tends to infinity then y tends to infinity* " and means that as x is given ever increasing values, so the values of y increase more and more.

The symbol $\to 0$ is used in the same way.

Thus as $x\to\infty$, $\dfrac{1}{x}\to 0$.

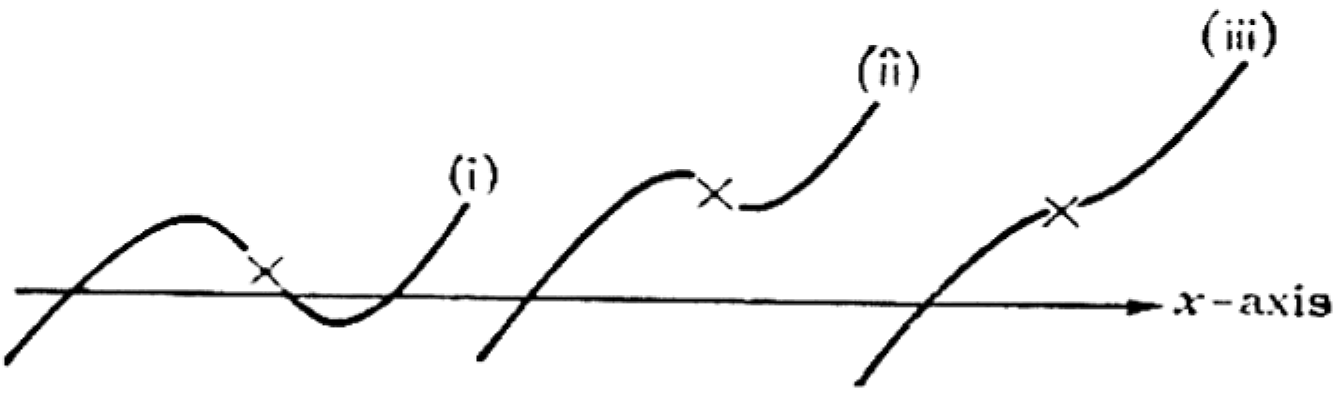

FIG. 13

The graphs, Fig. 13, (i), (ii), (iii), show the cases of

(i) three real roots and two real turning points,

(ii) one real root and two real turning points,

(iii) one real root and no turning points.

These curves are called " cubical parabolas ".

Note that straight lines can always be drawn to cut any such curves in three (real) points.

In each case there is a point of inflexion (marked by a cross in the figures); a striking property of such curves is that the point of inflexion is the centre of the curve and the curve is symmetrical about it, an instance of which is that the point of inflexion is the mid-point of the line joining the two turning points. An indication of this is given by the fact that the sum of the roots of the equation determining the x-coordinates of the turning points is $-2b/3a$ and the point of inflexion is at $x = -b/3a$. (See Examples 30, No. 5.)

A special case of such curves is $y = x^3$, in which the turning points and the point of inflexion coincide. (Fig. 14.)

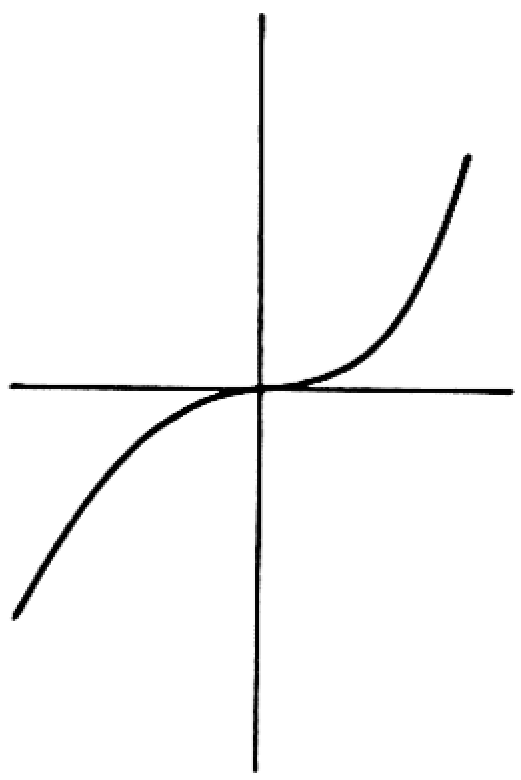

FIG. 14

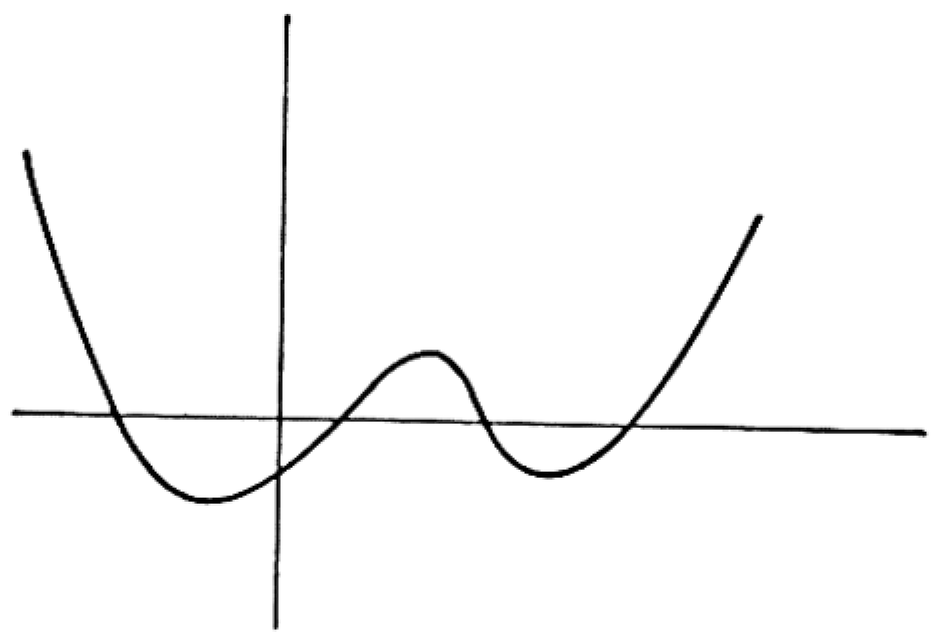

FIG. 15

Curves given by $y^2 = ax^3 + bx^2 + cx + d$ will

(i) be symmetrical about the x-axis since $(-y)^2 = y^2$,

(ii) lie only in those parts of the plane where the curve given by

$$y = ax^3 + bx^2 + cx + d \text{ is above the } x\text{-axis.}$$

These curves are called " semi-cubical parabolas ".

Graphs of curves of higher degree than the cubic may be considered in the same way, though they become increasingly more difficult.

The general function of the fourth degree $y=ax^4+bx^3+cx^2+dx+e$ with a positive may have the form $y=a(x-p)(x-q)(x-r)(x-s)$ with a graph as shown in Fig. 15; it has three (real) turning points and two points of inflexion.

Moving the curve downwards parallel to the y-axis (a change of e in y) will give a curve which meets the x-axis at least twice and perhaps four times (as in the figure), while moving it upwards will ultimately give a curve which does not cross the x-axis, indicating that an equation of the fourth degree may not have any real roots.

$\frac{dy}{dx}$ will be a cubic in x, and so the graph of such a quartic must have at least one real turning point; again $\frac{d^2y}{dx^2}$ will be quadratic, which shows that there are either two real or two coincident or *no* points of inflexion.

In the case of $y=f(x)$ where $f(x)$ is a polynomial in x of odd power, then the curve will always cross the x-axis at least once, but if the degree of $f(x)$ is even, then the curve may not cross the axis at all, or else it crosses it an even number of times.

Examples 30

1. Draw the graphs of:

   (i) $y=x(x-2)(x-3)$; (ii) $y=(x+1)(x^2+1)$;
   (iii) $y=(x+1)(x-1)^2$; (iv) $y=x^2(x-2)$.

2. Draw with 2 inches as unit on each axis, the graphs between $x=-1$ and $x=+1$ of $y=x^2$, $y=x^3$ and $y=x^4$.

3. Find the x-coordinate of the point of inflexion for the curve
$$y=(x-p)(x-q)(x-r).$$

4. Which of the following curves has real turning points?

   (i) $y=2x^3-15x^2+24x+6$; (ii) $y=2x^3-15x^2+45x+12$.

5. Show that by writing $x=x'-p$ and $y=y'-q$ the general cubic curve's equation (given on p. 63) can be put in the form
$$y'=ax'^3-rx'$$
provided $p=\frac{b}{3a}$ and $q=-(2b^3-9abc+27a^2d)/27a^2$.

   Find the value of r.

   [The shape of the curve is settled by the value of $\frac{r}{a}$. The new origin is the centre of the curve, and is the point of inflexion.]

6. Draw the graphs of:

(i) $y=(x+1)(x-2)(x^2+1)$; (ii) $y=x^3(x-2)$;
(iii) $y=x^3(x-2)(x-4)$; (iv) $y=x^3(x^2-1)$;
(v) $y=x^2(x-2)^2(x-3)$; (vi) $y=x^2(x-2)^2(x-3)^2$.

7. Show that the graphs (iii) and (iv) in No. 6 each have two real turning points.

8. Draw the graphs of:

(i) $y^2=x(x-2)(x-3)$; (ii) $y^2=(x+1)(x-1)^2$;
(iii) $y^2=x^2(x-2)$; (iv) $y^2=x^3$.

9. Draw the graphs of:

$$y=x(x-1)(x-2) \text{ and of } y^2=x(x-1)(x-2)$$

and show that the loop of the second curve between $x=0$ and $x=1$ goes above the first curve; also show that the line $x=2$ is the tangent to the second curve where $x=2$.

10. Show that the tangent to $y^2=(x-2)^3$ where $x=2$ is the line $y=0$.

Fractional Functions

Curves of type $\dfrac{\textbf{Quadratic in x}}{\textbf{Linear in x}}$.

If
$$y=\frac{ax^2+bx+c}{px+q},$$

then as $x \to -\frac{q}{p}$, $y \to \infty$, and hence

the line $px+q=0$ is said to be a vertical asymptote to the graph.

If on division $\dfrac{ax^2+bx+c}{px+q}=lx+m+\dfrac{n}{px+q}$,

the line $y=lx+m$ is an asymptote to the graph since as

$$x \to \infty, \frac{n}{px+q} \to 0 \text{ and } y \text{ approaches the value of } lx+m.$$

The graph is a *hyperbola*, with one asymptote parallel to the y-axis.

Example 1. Draw the graph of $y=\dfrac{x^2+x-1}{2(x-1)}$. ....(i)

By division $y=\frac{1}{2}x+1+\dfrac{1}{2(x-1)}$. ....(ii)

$\therefore$ the asymptotes are $x-1=0$ and $y=\frac{1}{2}x+1$.

The curve cuts the axis where $x^2+x-1=0$ or $x=\dfrac{-1 \pm \sqrt{5}}{2}$,

$$\simeq \cdot 62 \text{ or } -1 \cdot 62.$$

Also we have

x	-1	0	2
y	$\frac{1}{4}$	$\frac{1}{2}$	$2\frac{1}{2}$

Again, $\dfrac{dy}{dx}=\dfrac{1}{2}-\dfrac{1}{2(x-1)^2}$ using (ii).

$\therefore$ the turning points are where $(x-1)^2=1$, i.e. $x=2$ or 0, and so are the points L $(2, 2\frac{1}{2})$ and M $(0, \frac{1}{2})$ in Fig. 16.

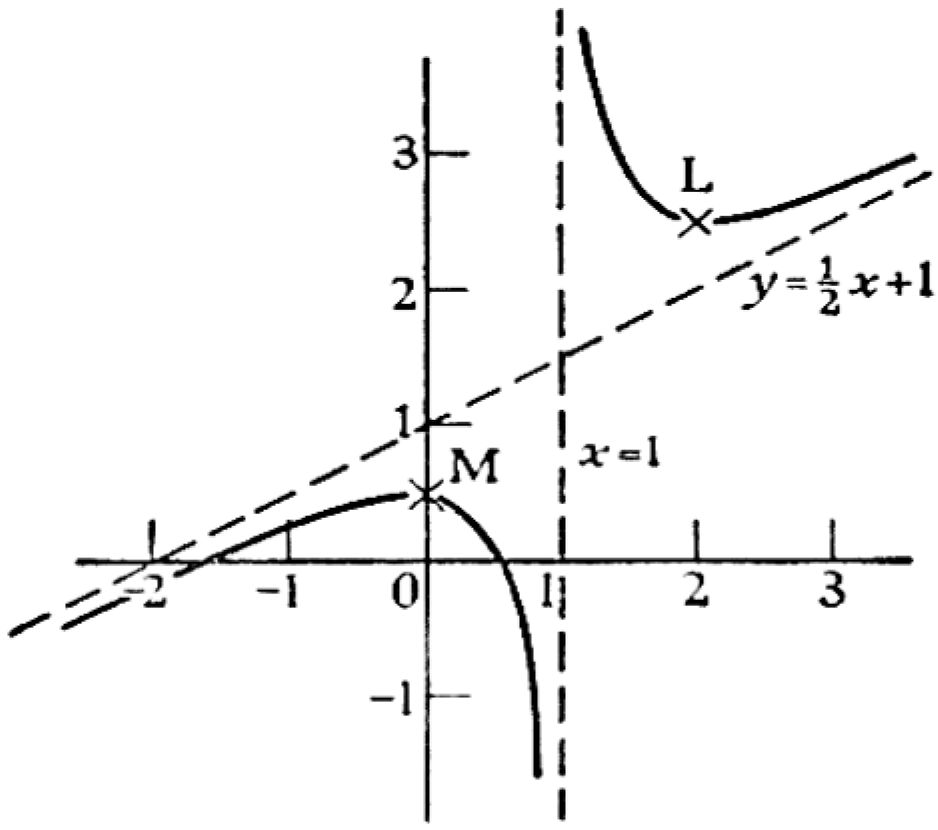

FIG. 16

$\therefore$ $y=2\frac{1}{2}$ is a minimum value for one branch and $y=\frac{1}{2}$ a maximum value for the other.

Thus y never lies between $\frac{1}{2}$ and $2\frac{1}{2}$.

This result can be found otherwise; if y is given, the equation for x is

$$x^2-x(2y-1)+2y-1=0.$$

This has (real) roots if $(2y-1)^2-4(2y-1)$ is positive, i.e. if $(2y-1)(2y-5)$ is positive, which is so for all values of y except those between $\frac{1}{2}$ and $2\frac{1}{2}$.

Examples 31. Draw the graphs of the following functions, finding the asymptotes and the maxima and minima for y:

1. $y=\dfrac{x^2-3x+4}{x-2}$.
2. $y=\dfrac{-x^2+4x-6}{2(x-1)}$.
3. $y=\dfrac{x^2+5x+8}{2(x+2)}$.
4. $y=\dfrac{12+x-x^2}{x+1}$.
5. $y=\dfrac{x^2}{x-1}$.
6. $y=\dfrac{6-x-x^2}{x}$.

Curves of type $\dfrac{\text{Quadratic in x}}{\text{Quadratic in x}}$.

These curves are of three kinds:

(i) those in which y is always finite;
(ii) those in which y can take all values from $-\infty$ to $+\infty$;
(iii) those in which y is restricted, in that there are two values between which it cannot lie but otherwise can take all values.

An example of each of these kinds will be given.

All of them have in common the following properties:

(*a*) If x is given, there is one and only one value for y.
(*b*) If y is given, the equation for x is a quadratic, so that there are two points in general on each parallel to the x-axis.
(*c*) The maximum and minimum values of y (if any) may be found by considering whether the equation for x has (real) roots. If this is done, the corresponding values of x can be found afterwards.

Alternatively, if the values of x where $\dfrac{dy}{dx}=0$ are found, the values of y can be found afterwards.

Example I. Draw the graph of $y=\dfrac{(2x+1)(x-3)}{x^2+1}$.

$y=0$ when $x=-\frac{1}{2}$ and when $x=3$, shown by points A and B in Fig. 17. When $x=0$, $y=-3$, shown by C.

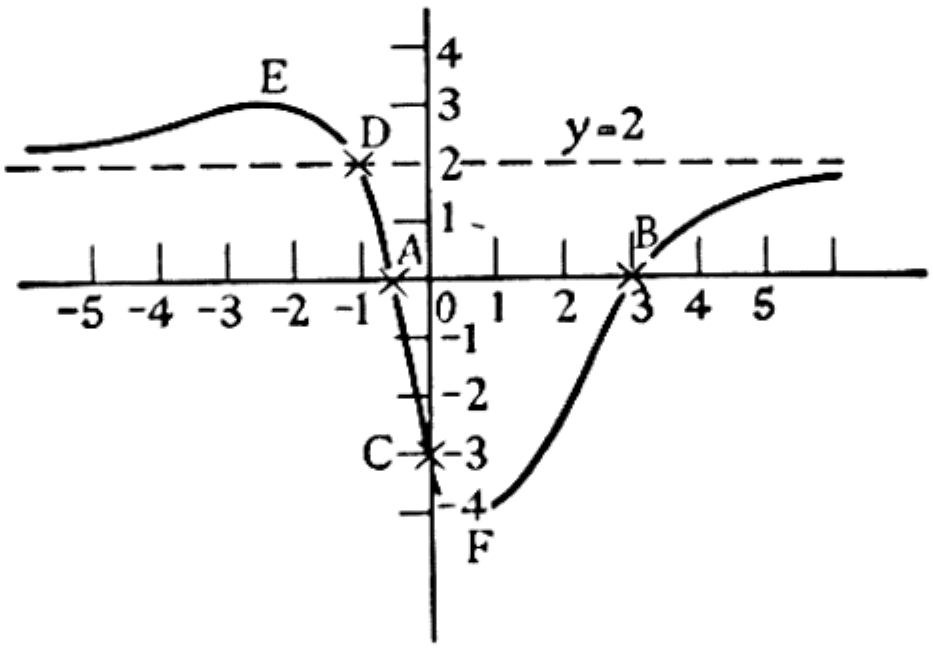

FIG. 17

There are no values of x making the denominator (x^2+1) zero, and so there are no " infinities ".

Since $$y=\frac{2x^2-5x-3}{x^2+1}=2-\frac{5x+5}{x^2+1},\quad y\to 2 \text{ as } x\to\pm\infty,$$

and so the graph has a (horizontal) asymptote $y=2$.

Moreover, when x is large and positive y is a little less than 2, while if x is large and negative y is a little greater than 2.

The curve meets $y=2$ where $5x+5=0$, i.e. where $x=-1$, shown by D.

The equation may be written as $(x^2+1)y=2x^2-5x-3$, i.e. as

$$x^2(y-2)+5x+(y+3)=0,$$

and this has real roots in x providing

$$25-4(y-2)(y+3)\geqslant 0\text{ ; i.e. if } 4y^2+4y-49\leqslant 0.$$

This reduces to $(y+\frac{1}{2})^2\leqslant 12\frac{1}{2}$, and so the turning points given by the equality sign are $y\simeq 3{\cdot}04$ or $-4{\cdot}04$, and points on the curve must have y-coordinates lying between these values.

The x-coordinates of the turning points are best given by equating the gradient to zero, and readers familiar with the calculus will readily obtain the equation $x^2+2x-1=0$, leading to $x\simeq -2{\cdot}41$ or ${\cdot}41$. The point E is the maximum turning point and F is the minimum turning point.

Example II. $y=\dfrac{(x-2)(x-4)}{2(x+2)(x-3)}$.

$y=0$ when $x=2$ and $x=4$ shown by A and B (Fig. 18) (the zeros are 2 and 4).

When $x=0$, $y=8/(-12)=-\frac{2}{3}$, shown by C.

$y\to\infty$ as $x\to -2$ and as $x\to 3$ (the infinities are -2 and 3), and so $x=-2$ and $x=3$ are vertical asymptotes.

In this case the zeros and the infinities *interlace*; i.e. one of the zeros lies between the infinities.

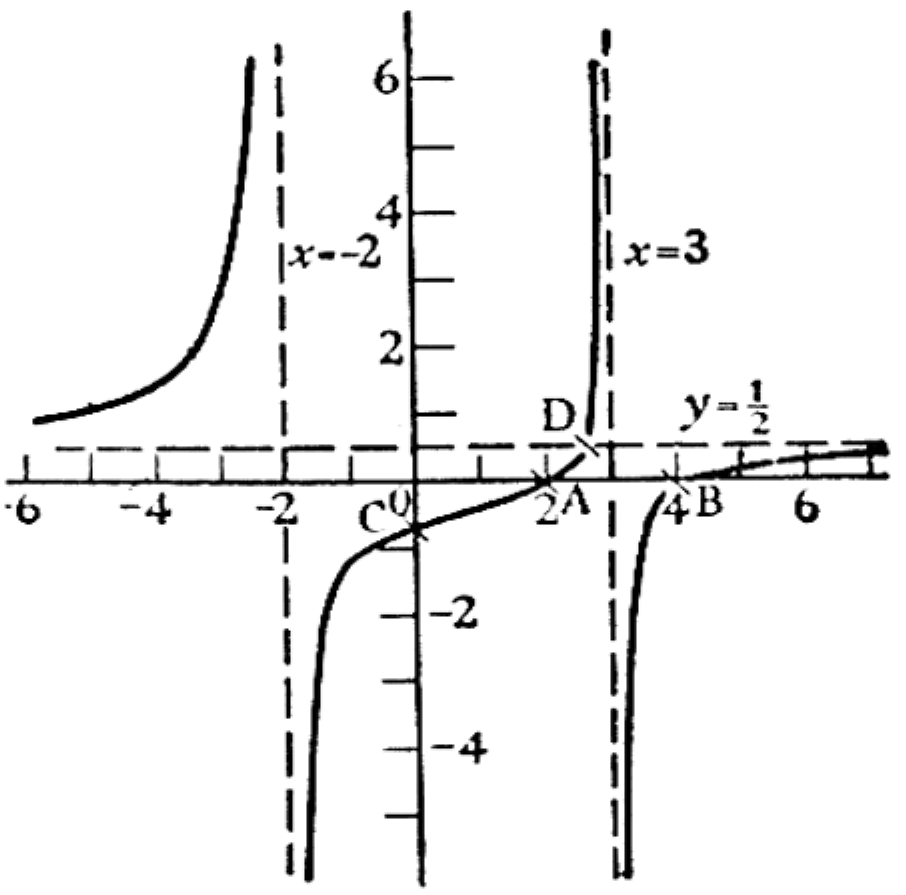

FIG. 18

Consider the signs of y for the various ranges of x as follows :

$-\infty < x < -2$, y is $\dfrac{(-)(-)}{(-)(-)}$, i.e. $+$.

$-2 < x < 2$, y is $\dfrac{(-)(-)}{(+)(-)}$, i.e. $-$, and in the same way we have :

for $2 < x < 3$, y is $+$; for $3 < x < 4$, y is $-$; for $4 < x < \infty$, y is $+$.

Since

$$y = \frac{(1-2/x)(1-4/x)}{2(1+2/x)(1-3/x)},$$

it is clear that for large x, $y \to \frac{1}{2}$ and so that $y=\frac{1}{2}$ is a horizontal asymptote, also when $y=\frac{1}{2}$ we have $x^2-6x+8=x^2-x-6$, i.e. $x=2{\cdot}8$ shown by D; and the curve does not cross this asymptote at any other point.

Again $2y(x^2-x-6)=x^2-6x+8$; i.e.

$$x^2(2y-1)-x(2y-6)-(12y+8)=0,$$

which is the equation whose roots determine the x of points with a given y. This equation has real roots if $(2y-6)^2+4(2y-1)(12y+8) \geqslant 0$, which reduces to $25y^2-2y+1 \geqslant 0$.

Therefore there will be points on the curve corresponding to a given y provided $(5y-\frac{1}{5})^2+\frac{24}{25} \geqslant 0$, which is true for *all* y ; this fact is seen from the graph, since it passes from $-\infty$ to $+\infty$ between $x=-2$ and $x=3$.

Example III. $y=\dfrac{2(x-2)(x-3)}{(x+1)(x-4)}$.

In this case the zeros are 2, 3 shown by A, B and the infinities are -1 and 4, and so both zeros lie within the infinities. $x=-1$ and $x=4$ are vertical asymptotes; also $y \to 2$ as $x \to \pm\infty$, and so $y=2$ is a horizontal asymptote (Fig. 19).

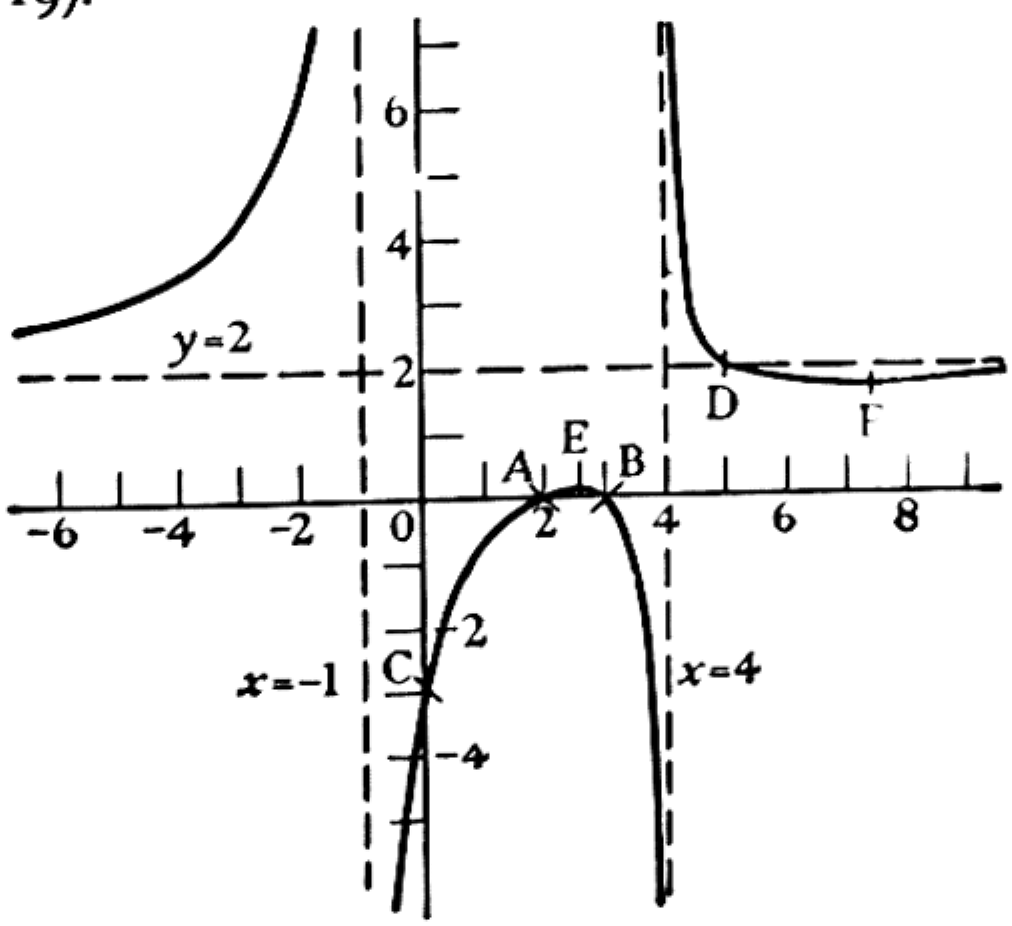

FIG. 19

When $x=0$, $y=12/(-4)=-3$, shown by C.

The signs of y passing along the x-axis from left to right are :

from $-\infty$ to -1 *positive*, from -1 to 2 *negative*, from 2 to 3 *positive*, from 3 to 4 *negative*, and from 4 to $+\infty$ *positive*.

The curve meets the horizontal asymptote $y=2$ where

$$x^2-5x+6=x^2-3x-4,$$

i.e. at $x=5$ shown by D.

Since $y(x^2-3x-4)=2(x^2-5x+6)$,

i.e. $x^2(y-2)-x(3y-10)-(4y+12)=0$,

there will be real x's corresponding to a given y providing

$$(3y-10)^2+4(y-2)(4y+12)\geqslant 0,$$

which reduces to $25y^2-44y+4\geqslant 0$. Consequently y cannot lie between the roots of $25y^2-44y+4=0$, which are $\simeq 1{\cdot}7$ or $\cdot 1$.

The x-coordinate of the turning points are given by putting $\dfrac{dy}{dx}$ zero, which may be shown to give the equation $x^2-10x+19=0$, and so $x\simeq 7{\cdot}45$ or $2{\cdot}55$. There is therefore a maximum turning point at E $(2{\cdot}55, \cdot 1)$ and a minimum turning point at F $(7{\cdot}45, 1{\cdot}7)$.

Examples 32

Draw the graphs of the following :

1. $y=\dfrac{3(x-2)(x-5)}{5(x+1)(x-3)}$.
2. $y=\dfrac{x(x-2)}{(x+1)(x-3)}$.
3. $y=\dfrac{(x+2)(x-4)}{(x+1)(x-3)}$.
4. $y=\dfrac{2(x-2)(x-3)}{2x^2+x+1}$.
5. $y=\dfrac{2x-5}{x(x+2)}$.
6. $y=\dfrac{x-2}{x^2+2}$.
7. Show that $\dfrac{kx^2+3x-4}{k+3x-4x^2}$ can assume all real values, when x is real, provided $1<k<7$.

   Draw a rough graph of the function when $k=1$. (B.)
8. If $y=\dfrac{x^2-x+1}{x^2+x+1}$ show that the quadratic equation to find x for a given value of y has real roots if $3y^2-10y+3$ is negative or zero, and determine for what range of values of y this condition is satisfied. (B.)
9. Draw the graph of the expression in No. 8.
10. Obtain the conditions satisfied by a, b, c if the quadratic expression ax^2+bx+c is positive for all real values of x.

    Show that if $y=\dfrac{3x^2+5x-12}{x^2-3x+2}$, then y can assume any real value between $-\infty$ and $+\infty$ for real values of x.

Draw a rough graph of the function, indicating clearly how it behaves for numerically large values of x. (S.)

11. If $f(x)=\dfrac{12x}{x^2+4}$, prove that if x is real, $f(x)$ must lie between -3 and $+3$.

Draw the graph of $y=\dfrac{12x}{x^2+4}$ from $x=-4$ to $x=4$.

On the same diagram draw the graph of $y=4x-x^2$ from $x=-1$ to $x=5$. Hence show that the equation

$$x^3-4x^2+4x-4=0$$

has only one root and find its approximate value. (L.)

12. Prove that if $y=\dfrac{(x+1)(x-3)}{(x-1)^2}$ there are two real values of x for each real value of y less than 1 and that these values are equidistant from 1.

Sketch the general form of the graph. (L.)

13. For the curve given by

$$y=\frac{y_2(x-x_1)^2+ky_1(x-x_2)^2}{(x-x_1)^2+k(x-x_2)^2}$$

where $y_1>y_2$ show that if k is positive y lies between y_1 and y_2, while if k is negative y cannot lie between y_1 and y_2.

Show further that the turning points are (x_1, y_1) and (x_2, y_2), and explain which of these points is the maximum point for y.*

14. Show that plotting values of (x, y) determined by the values of x suggested gives an insufficient guide to the shape of the curve in the following cases:

(i) $y=x^3$ using $x=0, \pm 1$;
(ii) $y=x^3(10-x^2)$ using $x=0, \pm 1, \pm 3$;
(iii) $4y=x^2(x+3)(x^2-5)$ using $x=-3, \pm 1, \pm 1, \pm 2$.

Graphs of Expressions involving the Modulus Sign

The notation $|x|$ is used to mean the numerical value of x; for instance, if x is 5 or -5, the value of $|x|$ is 5. It is called the *modulus* of x.

The graph of an expression such as $|x|$ or $|(x-1)(x-3)|$, when the modulus sign is applied to a single function of x, is not appreciably more difficult than the graphs without the modulus sign, viz. those of x and $(x-1)(x-3)$.

* In *Mathematical Gazette*, Dec. 1955, C. V. Gregg gives other properties of these curves.

All that is needed for these two graphs is to draw the graphs of x and $(x-1)(x-3)$ and replace the parts below the x-axis by their reflections in that axis. (Figs. 20, 21.)

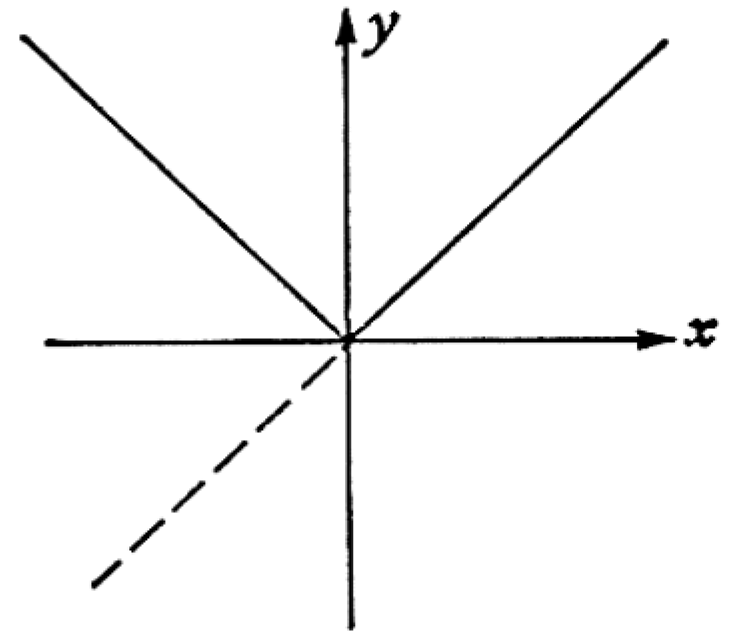

FIG. 20
Graph of $y=|x|$
[The dotted line, part of $y=x$, is reflected in the x-axis]

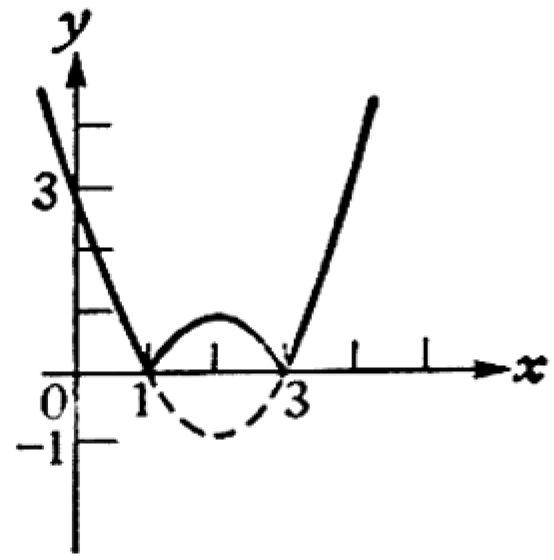

FIG. 21
Graph of $y=|(x-1)(x-3)|$
[The dotted line, part of $y=(x-1)(x-3)$, is reflected in the x-axis]

If the modulus of a function involving both x and y has a constant value, a double graph is obtained.

Thus the equation $|y-x^2|=2$ is equivalent to the two equations $y-x^2=2$ and $y-x^2=-2$. Thus the graph consists of two parabolas obtained by shifting the graph of $y-x^2=0$ a distance 2 parallel to the y-axis both upwards and downwards.

Similarly the graph of $|y-x|=1$ consists of two lines parallel to $y-x=0$, namely the lines $y-x=1$ and $y-x=-1$.

Some examples will be given of cases where the modulus sign is applied to more than one function. In such cases it is often desirable to divide the plane into several regions which are considered separately.

Example I

Draw the graph of $y=|y-3x|+|x-2|$.

The two lines $y-3x=0$ and $x-2=0$, shown by dotted lines in Fig. 22, divide the plane into the four regions A, B, C, D.

In region A, $y-3x$ is positive and $x-2$ is negative, and so

$$y=y-3x+2-x, \text{ i.e. } x=\tfrac{1}{2}.$$

This gives the position of the line $x=\frac{1}{2}$ above P $(\frac{1}{2}, \frac{3}{2})$.

In B, $y-3x$ is negative and $x-2$ is also negative.

$\therefore$ in B,
$$y=3x-y+2-x,$$
$$\text{i.e. } y=x+1, \text{ giving the line } PQ.$$

In C, $y-3x$ is negative and $x-2$ is positive, and so $y=3x-y+x-2$,

i.e. $y=2x-1$, giving the line QR.

In D, both $y-3x$ and $x-2$ are positive, and so $y=y-3x+x-2$. This gives $x=-\frac{1}{2}$, which has no part in the region.

Therefore the locus is made up of the portions of the three lines shown by continuous lines.

Example II

Draw the graph given by $|y-x^2|=|y^2-x|$.

The two dotted graphs in Fig. 23 are those of $y-x^2=0$ and $y^2-x=0$, and they divide the plane into the five regions A, B, C, D, E.

In A and B $y-x^2$ is positive, while in C, D, E it is negative.

In A, C, E y^2-x is positive, and in B, D it is negative.

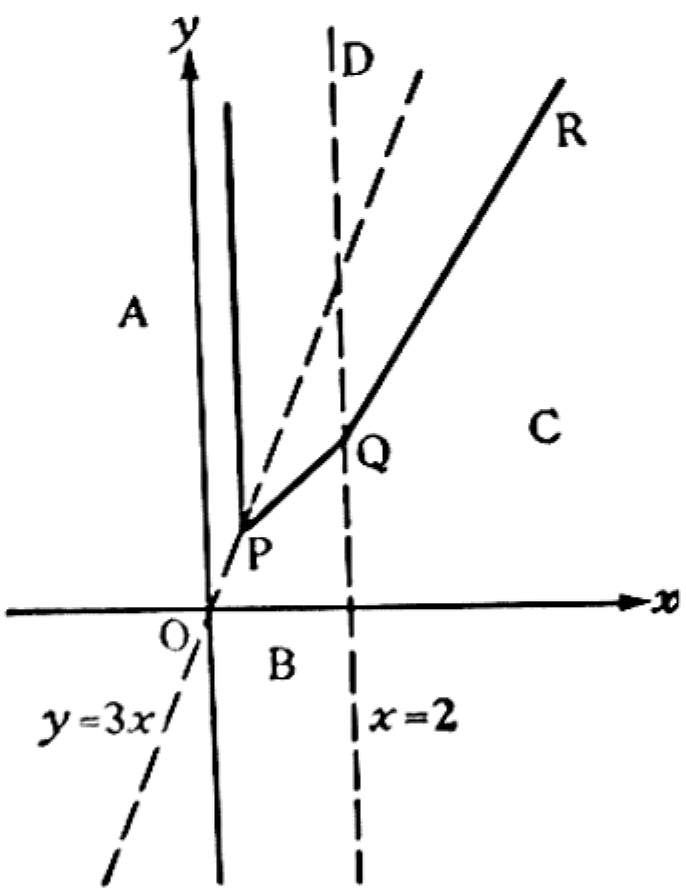

FIG. 22

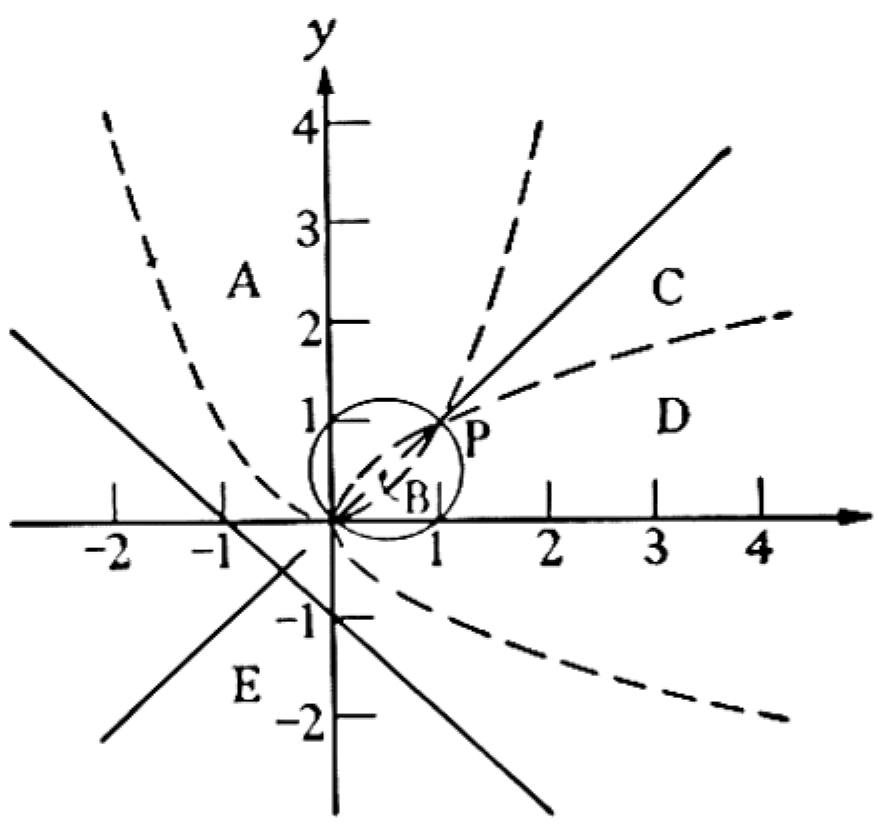

FIG. 23

Therefore in regions A and D $y-x^2$ and y^2-x have the same sign, and for these regions $y-x^2=y^2-x$; i.e. $x^2+y^2-x-y=0$, which is the equation of the circle on OP as diameter.

In regions B, C, E $y-x^2$ and y^2-x have opposite signs, and for these regions $y-x^2=-(y^2-x)$, i.e. $y^2-x^2+y-x=0$, or $(y-x)(y+x+1)=0$, which is the combined equation of the lines $y=x$ and $y+x=-1$.

The required locus is made up of these two lines and the circle.

Example III

Draw the graph given by $|y-x^2|=|2y-2x-1|$.

Here we sketch the graphs (Fig. 24) of $y-x^2=0$ and $2y-2x-1=0$, a parabola and a line (dotted). These divide the plane into five regions, A, B_1, B_2, C, D, of which A, B_1 and B_2 are outside the parabola while C and D are inside; also B_1, B_2 and C are above the line and A and D below it.

In C both $y-x^2$ and $2y-2x-1$ are positive; in A they are both negative.

In these cases the equation reads

$$y-x^2=2y-2x-1 \quad \text{or} \quad y=1+2x-x^2=2-(x-1)^2.$$

In B_1, B_2 and in D the expressions $y-x^2$ and $2y-2x-1$ have opposite signs; so for these regions the equation reads

$$y-x^2=1+2x-2y \quad \text{or} \quad 3y=1+2x+x^2=(1+x)^2.$$

Thus we arrive at two parabolas; the whole of the first lies in A or C and the whole of the second in B_1, B_2 or D; thus the whole of each parabola belongs to the (double) locus.

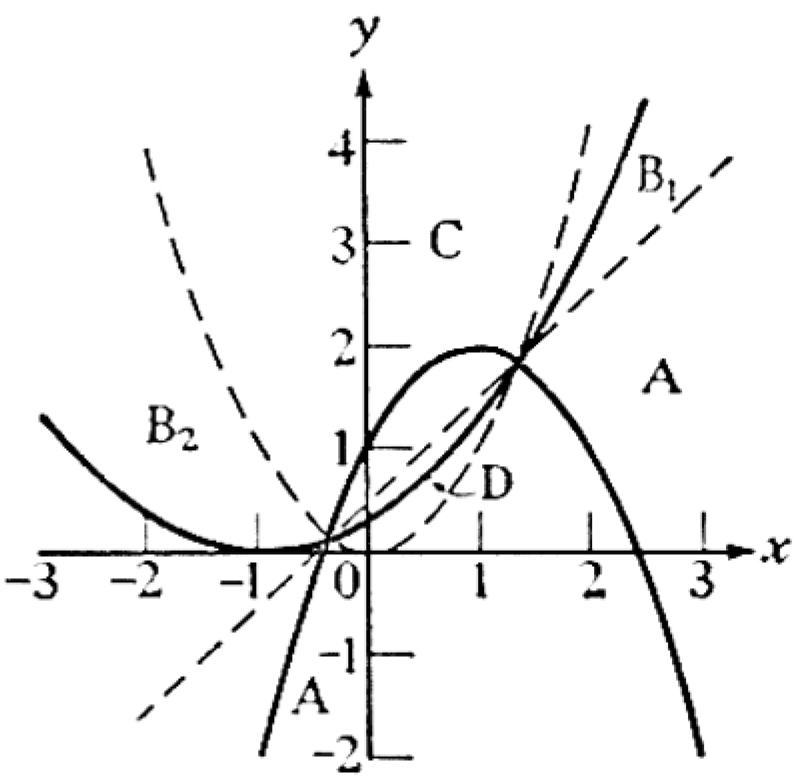

FIG. 24

The two guide graphs are shown dotted and the required locus in firm line.

Note that the points of intersection are the same for both (as they must be).

For such functions involving the modulus sign, note that there may be points where the gradient changes abruptly (Example I) or there may be for any value of x two different gradients (Example III).

Thus the functions have not got a derived function in the usual sense.

Examples 33. Draw the graphs given by the following equations:

1. $y=|x^2-9|$.
2. $y=|x^2-x-6|$.
3. $y=|(x+2)(x-1)(x-4)|$.
4. $|x+y|=2$.
5. $|x-2x^2|=1$.
6. For what value of q does $|y-x^2|=q$ not give more than one curve?

Draw graphs of:

7. $y=$ [illegible] $+|x|$.
8. $y=|2x-3|-|x|$.
9. $2y=|x-2|+|x-y|$.
10. For $y=|2x|+|y-x^2|$ show that parts of three parallel lines are obtained, and two parts of parabolas.
11. Show that $|x^2+y^2-x-y|=|y-x|$ gives two complete circles.
12. Show that $|y-x^2|=|y^2-x|$ gives a circle and two straight lines.
13. Show that $|x^2+y^2-y|=|y-x^2|$ gives the x-axis and an ellipse.
14. Draw the graphs of

(i) $y=|2x|-|1-x|$; (ii) $2y=|x-1|+|x-y|$.

(N.U.J.B.)

The Solution of Numerical Equations

In the case of quadratic equations approximate answers, when necessary, are found after the accurate answers have been stated in terms of square roots; e.g. if $x^2-6x=4$, then $(x-3)^2=13$;

$$\therefore\ x=3\pm\sqrt{13},$$

and so $x \simeq 3+3{\cdot}61$ or $3-3{\cdot}61$, i.e. $6{\cdot}61$ or $-{\cdot}61$ to 2 decimal places.

For equations of higher degree approximations begin at once; often a graph is used to locate the roots roughly and then various methods employed to locate them more accurately.

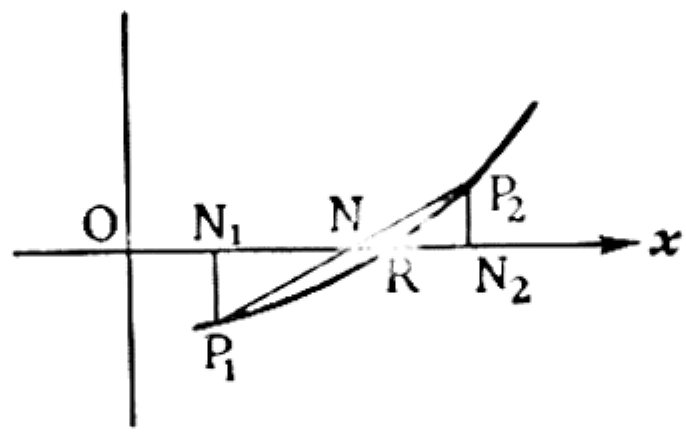

FIG. 25

Suppose the equation is $f(x)=0$ and the graph $y=f(x)$ is drawn (Fig. 25), then if $ON_1\,(\equiv x_1)$ is a value of x for which $N_1P_1=f(x_1)$ is negative and if $ON_2\,(\equiv x_2)$ is such that $N_2P_2=f(x_2)$ is positive, it is clear that a curve joining P_1 to P_2 must cross the x-axis at some point R between them.

[The curve may cross the x-axis an odd number of times, but if N_1 and N_2 are close enough the curve will only cross once.]

x_R is the root of $f(x)=0$ between x_1 and x_2 and if the line P_1P_2 meets the x-axis in N, then R lies between N and N_2 if the graph is as sketched, with $f(x_N)$ negative. The reader should draw the diagram in which R is between N_1 and N, in which case $f(x_N)$ will be positive.

The position of N is found by making use of the similar $\triangle$s P_1NN_1 and P_2NN_2 to give $NN_1=\dfrac{P_1N_1}{P_1N_1+N_2P_2}\cdot N_1N_2$.

Example I. Find the largest positive root of $f(x)\equiv x^4-5x+2=0$ correct to two decimal places.

From the table

x	0	1	2	3
$f(x)$	2	-2	8	68

,

and the fact that $f'(x)=4x^3-5$ is positive for all values of $x\geqslant 2$ it is clear that the root required lies between 1 and 2.

Now $f(1{\cdot}5)\simeq 5{\cdot}06-7{\cdot}5+2=-{\cdot}44$ which, being negative, shows that the root lies between $1{\cdot}5$ and 2, and nearer to $1{\cdot}5$.

$f(1\cdot 6) \simeq 6\cdot 55 - 8 + 2 = +\cdot 55$; hence the root lies between $1\cdot 5$ and $1\cdot 6$, and making use of the figure above with $ON_1 = 1\cdot 5$ and $ON_2 = 1\cdot 6$ we have

$N_1N = \dfrac{\cdot 44}{\cdot 44 + \cdot 55}$ of $\cdot 1 \simeq \cdot 04$ and the root is approximately $1\cdot 54$.

$$\left.\begin{aligned} f(1\cdot 54) &\simeq 5\cdot 62 - 7\cdot 70 + 2 = -\cdot 08 \\ f(1\cdot 55) &\simeq 5\cdot 78 - 7\cdot 75 + 2 = +\cdot 03 \end{aligned}\right\} \quad \therefore \text{ the root lies between } 1\cdot 54 \text{ and } 1\cdot 55 \text{ and is nearer } 1\cdot 55.$$

Newton's Method of Approximation

This is a method which makes use of the tangent at a point near the root.

Suppose the equation is $f(x) = 0$; the graph of $y = f(x)$ is drawn (Fig. 26), then if $ON (= x_1)$ is a value of x for which $f(x)$ is fairly small, then x_1 is a first approximation to the root.

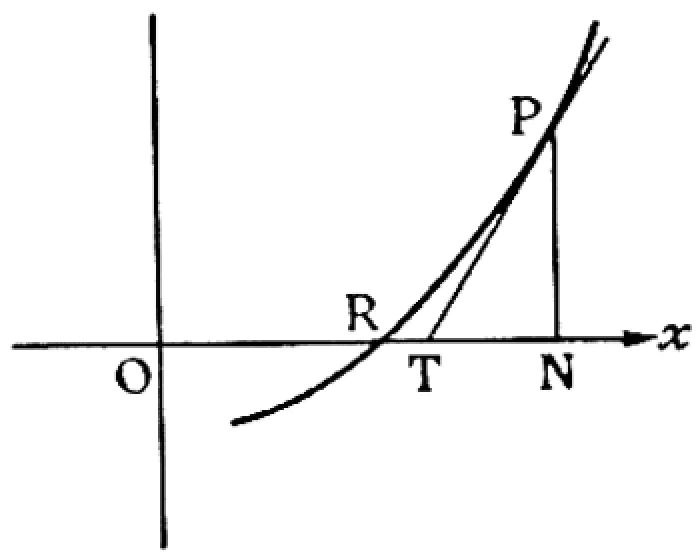

FIG. 26

If P is the point on the graph given by x_1 so that $NP = f(x_1)$, and if the tangent at P cuts the axis at T, then T will (nearly always) be nearer than N is to R, where OR is the required root, so that OT is a better approximation to the root than ON.

Now $TN = NP \div$ gradient at $P = f(x_1)/f'(x_1)$ in the notation of the Calculus.

Thus we get the rule that if x_1 is an approximation to the root, then $x_1 - f(x_1)/f'(x_1)$ is a better approximation. (I)

Calling OT x_2, we can repeat the process and get $x_2 - f(x_2)/f'(x_2)$, which will be still nearer to the root OR.

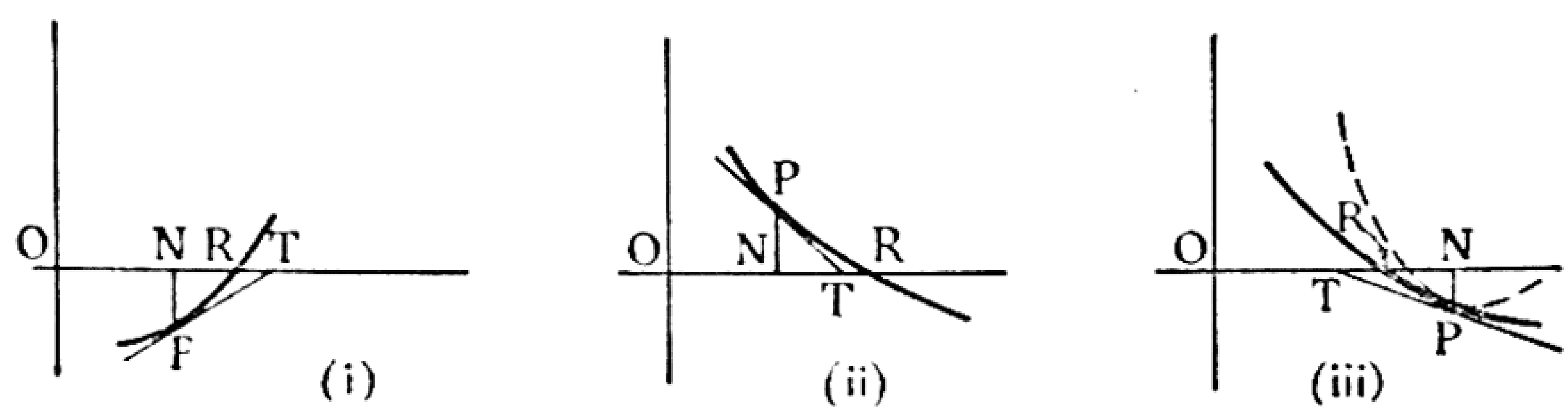

FIG. 27

It will be seen that the position of T is always given by

$$x_1 - f(x_1)/f'(x_1).$$

Thus in Fig. 27 (i) $OT = ON + NT$, but $f(x_1)$ being negative, the *length* of PN is $-f(x_1)$ and NT which is positive $= -f(x_1)/f'(x_1)$ for $f'(x_1)$ is positive.

That the same result is true for Fig. 27 (ii), (iii), is left as an example to be worked (Examples 34, No. 1).

Fig 27 (iii) shows that it is not certain that T will be nearer the root than N, for if the curve were as that shown dotted, N would be the nearer of the two.

Root between two numbers

It is often convenient to take two numbers between which the root lies, as in Example I, and to use Newton's method for each.

The root lies between $x_1 (\equiv ON_1)$ and $x_2 (\equiv ON_2)$—Fig. 28. We can find T_1 and T_2 if P_1T_1 and P_2T_2 are tangents by Newton's formula.

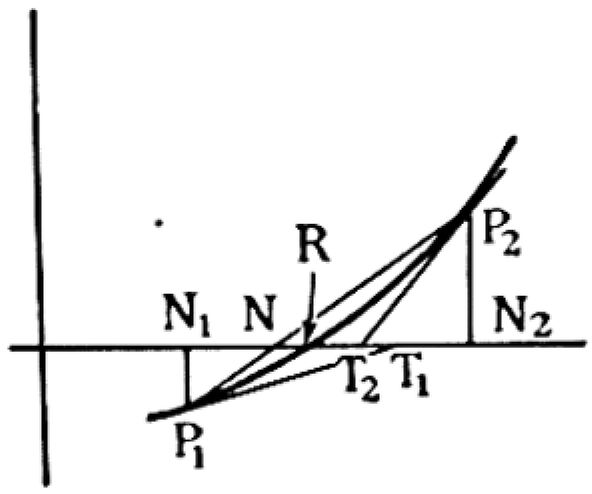

FIG. 28

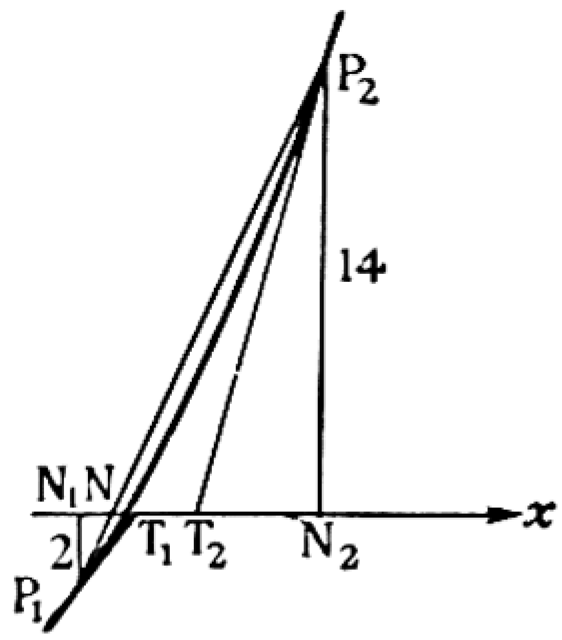

FIG. 29

Also the point N where chord P_1P_2 cuts the axis is found using the similar $\triangle$s as before.

The range in which the root lies is reduced in this way from N_1N_2 to NT_1 or NT_2 whichever is the smaller.

Example II. Find the largest positive root of $f(x) \equiv x^3 - 3x^2 - 2 = 0$.

The table

x	0	1	2	3	4	5
$f(x)$	-2	-4	-6	-2	14	48

shows a root between $x = 3$ and $x = 4$.

$f'(x) = 3x^2 - 6x$ and is positive for all $x \geqslant 4$, so the largest positive root lies between 3 and 4.

Now $f'(3) = 9$ and $\therefore\ N_1T_1 = \frac{2}{9}$; $x_{T_1} = 3\frac{2}{9}$ (Fig. 29).

$f'(4) = 24$; $\quad \therefore\ T_2N_2 = \frac{14}{24} \simeq \cdot 58$; $x_{T_2} = 3\cdot 42$.

Also $N_1N = \dfrac{2}{2+14} \cdot 1 = \frac{1}{8}$; $\quad \therefore\ x_N = 3\frac{1}{8}$.

$\therefore\ T_1$ is nearer to N than T_2 and the root lies between $3\frac{1}{8}$ and $3\frac{2}{9}$.

We could proceed towards a closer approximation using either of these values (3·125 or 3·222), but will choose to work from 3·2.

$$f(3{\cdot}2) = {\cdot}048, \quad f'(3{\cdot}2) = 11{\cdot}52,$$

and so the root is approximately $3{\cdot}2 - \dfrac{{\cdot}048}{11{\cdot}52} \simeq 3{\cdot}2 - {\cdot}004$, which gives 3·196 as the approximate root.

Example III. Find the roots of $f(x) \equiv x^3 - 3x + 1 = 0$ roughly, and the middle one with a possible error $\not> {\cdot}001$.

x - -	-2	-1	0	1	2
$f(x)$ - -	-1	3	1	-1	3
$f'(x) = 3x^2 - 3$	9	0	-3	0	9

The table shows that the roots lie in the intervals -2 to -1, 0 to 1, 1 to 2.

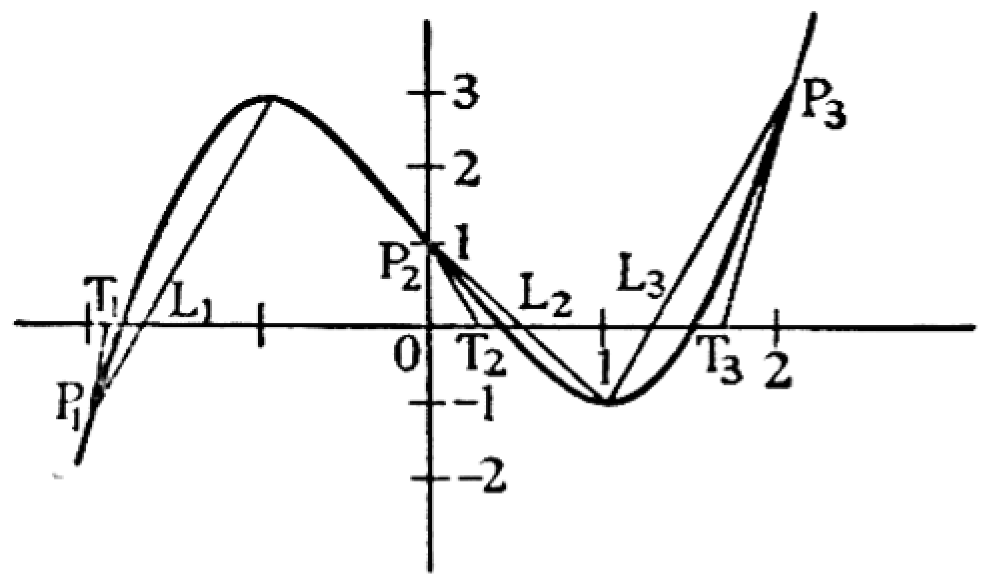

FIG. 30

In Fig. 30, $x_{T_1} = -2 + \frac{1}{9} = -1\frac{8}{9}$; $x_{L_1} = -2 + \frac{1}{4} = -1\frac{3}{4}$;

$x_{T_2} = 0 + \frac{1}{3} = \frac{1}{3}$; $x_{L_2} = 0 + \frac{1}{2} = \frac{1}{2}$;

$x_{T_3} = 2 - \frac{3}{9} = 1\frac{2}{3}$; $x_{L_3} = 1 + \frac{1}{4} = 1\frac{1}{4}$,

so the roots lie between $-1\frac{8}{9}$ and $-1\frac{3}{4}$, $\frac{1}{3}$ and $\frac{1}{2}$, $1\frac{1}{4}$ and $1\frac{2}{3}$.

To get the root between $\frac{1}{3}$ and $\frac{1}{2}$ it is natural to use $x = {\cdot}4$.

$$f({\cdot}4) = {\cdot}064 - 1{\cdot}2 + 1 = -{\cdot}136,$$

$$f(\tfrac{1}{3}) = \tfrac{1}{27} \simeq {\cdot}037\,; \quad f'(\tfrac{1}{3}) = -2\tfrac{2}{3}\,;$$

$\therefore$ in Fig. 31, $x_{T_2'} = \frac{1}{3} + \frac{1}{27}/2\frac{2}{3} = \frac{1}{3} + \frac{1}{72} \simeq {\cdot}347$,

$$x_{L_2'} = \tfrac{1}{3} + \frac{{\cdot}037}{{\cdot}037 + {\cdot}136}({\cdot}4 - {\cdot}3333)\,;$$

$$\simeq {\cdot}3333 + {\cdot}0143 = {\cdot}3476.$$

$\therefore$ the root lies between ·347 and ·3476 and can be given as ·347 with error $\not> {\cdot}001$.

Examples 34

1. Show that the formula $OT = x_1 - f(x_1)/f'(x_1)$ holds for Fig. 27 (ii) and Fig. 27 (iii).

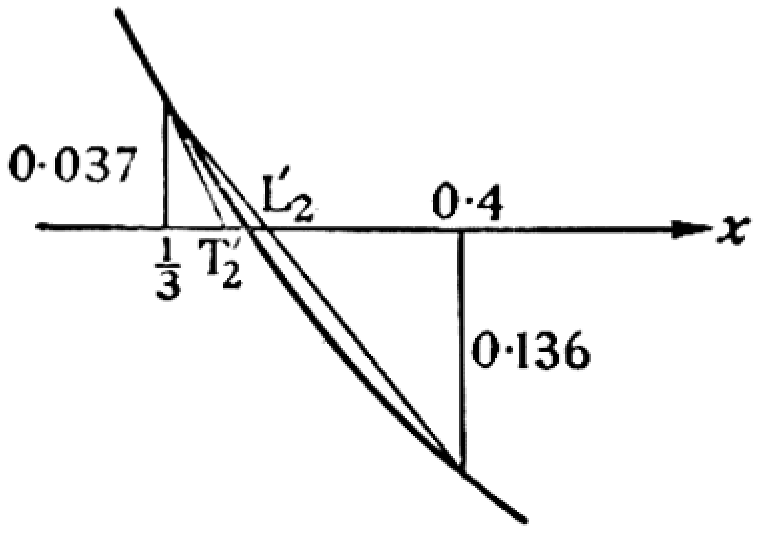

FIG. 31

2. Show that the root between 2 and 3 of the equation
$$x^3 - 2x^2 - 2x + 2 = 0$$
lies between 2·48 and 2·49.
3. Show that the root near $x = 1$ of the equation $x^4 - 50(x - 1) = 0$ $\simeq 1 \cdot 022$.
4. Show there is a root between $x = 2{\cdot}8$ and $x = 2{\cdot}9$ of the equation $2x^4 - 5x^2 - 9x - 60 = 0$ and find it to 3 significant figures.
5. Find the smaller positive root of $x^3 - 2x^2 - 2x + 2 = 0$. Find also the negative root. (Answers to 2 decimal places).
6. Find the root between $x = 2$ and $x = 3$ of $x^4 - 4x^3 - 13x^2 + 34x + 30 = 0$ correct to 3 significant figures.
7. Use Newton's method twice to find the root close to $x = 3$ of the equation $x^4 - 4x^3 - 34x^2 + 76x + 120 = 0$.
8. The equation $x^5 - x - 1 = 0$ has a root between 1 and 2; find it correct to 3 significant figures.
9. With θ measured in radians, solve the equation $2\theta = 25(1 - \cos\theta)$ approximately. First replace $\cos\theta$ by $\left(1 - \frac{\theta^2}{2}\right)$ to give a first approximation; secondly use Newton's method and this approximate root in the equation obtained by replacing $\cos\theta$ by $(1 - \theta^2/2 + \theta^4/24)$ to give the root to 4 decimal places.

CHAPTER IV

INDICES AND LOGARITHMS. EXPONENTIAL FUNCTIONS

Indices

When p and q are positive integers, the familiar index laws are:

(i) x^p is the shorthand notation for the product of p x's;

(ii) $x^p \times x^q = x^{p+q}$;

(iii) $x^p \div x^q = x^{p-q}$ if $p > q$;

(iv) $(x^p)^q = x^{pq}$.

As particular instances of these laws we have: x^4 means $x \times x \times x \times x$; $x^3 \times x^2 = x^{3+2} = x^5$; $x^7 \div x^2 = x^{7-2} = x^5$;

$$(x^2)^3 = x^2 \times x^2 \times x^2 = x^{2.3} = x^6.$$

We now proceed to give definitions of x^p when p is a fractional or a negative number (for (i) above cannot be applied), and these definitions are chosen so that the same index laws (ii), (iii), (iv) are obeyed.

$x^{\frac{1}{2}}$ obeys (ii) provided $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$.

Thus $x^{\frac{1}{2}}$ is a square root of x provided law (ii) holds for a fractional index. So we make this the definition for $x^{\frac{1}{2}}$ with the further restriction that $x^{\frac{1}{2}} = +\sqrt{x}$; thus $\pm 3^{\frac{1}{2}}$ is the same as $\pm\sqrt{3}$.

Similarly $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x$ and $x^{\frac{1}{3}}$ is defined as $\sqrt[3]{x}$.

Also $x^{\frac{3}{2}} \times x^{\frac{3}{2}} = x^{\frac{3}{2}+\frac{3}{2}} = x^3$, and so $x^{\frac{3}{2}} = \sqrt{x^3}$; in general $x^{p/q} = \sqrt[q]{x^p}$.

Again, using law (iv), $(x^{\frac{1}{2}})^3 = x^{\frac{3}{2}}$, and so $x^{\frac{3}{2}} = (\sqrt{x})^3$.

For negative indices to satisfy (ii), notice that $x^{-1} \times x^3 = x^{-1+3} = x^2$; and so x^{-1} must be defined as equal to $\frac{x^2}{x^3}$, i.e. $x^{-1} = \frac{1}{x}$.

Similarly $x^{-2} \times x^3 = x^{-2+3} = x$, i.e. $x^{-2} = \frac{x}{x^3} = \frac{1}{x^2}$; in general $x^{-n} = \frac{1}{x^n}$.

Notice the sequence of terms $\ldots, x^4, x^3, x^2, x^1, —, x^{-1}, x^{-2}, x^{-3}, x^{-4}, \ldots$.

Each term before the gap is obtained from the term in front by dividing by x, and this is also true for the terms after the gap since they are $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots$.

If the series of indices is to be complete, the gap must be filled by

x^0; or if the process of dividing each term by x to give the succeeding term is used, the gap must be filled by 1. This suggests defining x^0 as 1.

Again using law (i), $x^0 \times x^q = x^{0+q} = x^q$, and so x^0 is *defined* as 1.

Note that x^0 has no meaning if $x = 0$, nor has $x^{\frac{1}{2}}$ a meaning if x is negative; but $\sqrt[3]{(-8)} = (-8)^{1/3} = -2$.

Examples 35

1. Write down the values of:

(i) $4^{\frac{1}{2}}$; (ii) $25^{\frac{1}{2}}$; (iii) $32^{\frac{1}{5}}$; (iv) $27^{\frac{1}{3}}$;
(v) 6^{-1}; (vi) 9^{-2}; (vii) 4^0; (viii) $64^{-\frac{1}{2}}$;
(ix) $8^{-\frac{1}{3}}$; (x) $16^{\frac{1}{4}}$; (xi) $81^{-\frac{1}{4}}$; (xii) $125^{-\frac{1}{3}}$.

2. Write down the values of:

(i) $8^{\frac{2}{3}}$; (ii) $27^{-\frac{2}{3}}$; (iii) $64^{-\frac{5}{6}}$; (iv) $16^{-\frac{3}{4}}$;
(v) $100^{\frac{3}{2}}$; (vi) $16^{\frac{5}{4}}$; (vii) $4^{\frac{5}{2}}$; (viii) $16^{1\cdot5}$;
(ix) $32^{1\cdot2}$; (x) $16^{1\cdot25}$; (xi) $25^{-1\cdot5}$; (xii) $100^{-\cdot5}$.

3. Express in their simplest form with positive indices:

(i) $(a^9x^{15})^{\frac{1}{3}}$; (ii) $(a^7)^{-2}$; (iii) $(x^8)^{-\frac{3}{2}}$; (iv) $(16a^{12})^{\frac{3}{4}}$;
(v) $(8x^9)^{-\frac{2}{3}}$; (vi) $(9x^8)^{\frac{3}{2}}$; (vii) $(8x^{-9})^{\frac{2}{3}}$; (viii) $(9x^{-8})^{-\frac{3}{2}}$;
(ix) $(x^9y^{12})^{-\frac{1}{3}}$; (x) $\sqrt{(36a^8b^{-2})}$; (xi) $\sqrt[3]{(64x^{-6}y^6)}$;
(xii) $(-32x^{10}y^{-20})^{\frac{1}{5}}$.

Powers of 10

Perhaps some who are accustomed to working with logarithms and antilogarithms have not fully realised that they have been working with fractional powers of 10 and that the table of antilogarithms is a table giving the values of these powers, the fractions being expressed in decimals, to 4 places in a 4-figure table.

e.g. $\log 70 = 1\cdot8451$ means that $70 = 10^{1\cdot8451}$
and antilog $\cdot38 = 2\cdot399$ means that $10^{\cdot38} = 2\cdot399$.

Thus the theory of fractional indices, though it might not seem so at first, turns out to be one of the most practical items in Algebra.

Examples 36

1. Write as powers of 10, using tables: (i) 2; (ii) 3; (iii) $\sqrt{5}$; (iv) $7\sqrt{2}$; (v) $\sqrt[3]{6\cdot2}$; (vi) $6^{3/2}$.
2. Write without using the word logarithm (or log.):
(i) $\log 5 = \cdot6990$; (ii) $\cdot8 = \log 6\cdot31$; (iii) $\log 20 = 1\cdot3010$.
3. Write using the word logarithm (or log.):
(i) $10^{1\cdot6} = 39\cdot81$; (ii) $798 = 10^{2\cdot902}$; (iii) $4 = 2^2$.
Also write (i), (ii), (iii) using the word anti-log.

The Language of Logarithms

We can go from the statement $2=10^{\cdot 3010}$ to the alternative version, $\log 2=\cdot 3010$ if in the second statement it is understood that 10 is the *base* (as it is called) of the logarithms.

Logarithms to other bases are also used; for example, $8=2^3$ can be written "3 is the logarithm of 8 to the base 2" or more briefly $3=\log_2 8$.

Similarly $a^b=c$ is the same as $b=\log_a c$ (i.e. log c to base a), and in expressing the equation in this way we are using "the language of logarithms".

Note that the word logarithm (or log.) *is not attached to the logarithm itself*. If we say $\log 20=1\cdot 3010$ we mean that $1\cdot 3010$ is the logarithm of the number 20.

The language of logarithms needs practice. At first it is not clear that the statement $\log_4 64=3$ is equivalent to $4^3=64$, and that if $10^z=(x+1)^2$ then $z=2\log_{10}(x+1)$.

Examples 37

[The logarithms are to base 10 unless another base is mentioned.]

1. Write the numbers $\frac{1}{128}$ and $\sqrt{8}$ as powers of 2.
   Also state the logarithms to the base 2 of these numbers.
2. State the relation $\log 20=1\cdot 3010$ using the word "antilog".
   Also write it using indices.
3. Express in the language of logarithms:
   (i) $5^3=125$; (ii) $81=27^{4/3}$; (iii) If $k=a^4$ then $a^{12}=k^3$.
4. Express, using indices:
   (i) $\log_{13} 169=2$; (ii) $\frac{3}{2}=\log_{16} 64$; (iii) $\log_c a^{12}=6\log_c a^2$.
5. What is the logarithm of the cube of antilog $1\cdot 6$?
6. Find (i) the logarithm, (ii) the antilogarithm of $2\cdot 4771$.
7. What is the relation between x and y if $\log_4 t=x$ and $\log_8 t=y$?
8. Find x (i) if $\log_x 81=8$, (ii) if $\log_x 625=4/3$.
9. Express $\log \cdot 75=\bar{1}\cdot 8751$ using indices and not using the notation $\bar{1}$.
10. Find the logarithm of $10^{-\cdot 2}$ (i) to base 10, (ii) to base 3.

Laws of Logarithms

The laws for dealing with logarithms are deduced from the laws of indices.

If $p=a^x$ and $q=a^y$, it follows from the laws of indices that

$$pq=a^{x+y} \quad \text{and} \quad p/q=a^{x-y}.$$

In the language of logarithms these statements are :
If $\log_a p = x$ and $\log_a q, = y$ then $\log_a pq = x + y$.

Hence $\log_a pq = \log_a p + \log_a q$.

Similarly $\log_a p/q = x - y = \log_a p - \log_a q$.

Again if $x = \log_a p$, then $p = a^x$;

$$\therefore \log_a p^n = \log_a (a^x)^n = \log_a a^{xn}$$
$$= xn = n \log_a p.$$

Changing the Base of Logarithms

If $a = b^z$ and $N = a^x$, then $N = (b^z)^x = b^{zx}$.

In the language of logarithms, these equations are

$$z = \log_b a,\ x = \log_a N \text{ and } zx = \log_b N\ ;$$
$$\therefore \log_b N = \log_b a \,.\, \log_a N,$$

so that $\log_a N = \log_b N \div \log_b a$.

For example, $\log_3 5 = \log_{10} 5 \div \log_{10} 3$.

In the same way, from the statement that

$$\text{if } b^x = a \text{ then } b = a^{1/x},$$

we get $\log_a b = 1/\log_b a$.

Examples 38

1. Express in the language of logarithms : if $p = a^x$, $q = a^y$, $r = a^z$, then $pq \div r = a^{x+y-z}$.
2. What is the relation between x and y if $x = \log_9 5$ and (i) $y = \log_3 5$? (ii) $y = \log_3 125$?
3. Use tables to calculate $\log_3 5$ and $\log_5 3$.
4. Without using tables show that

   (i) $\log 800 - \log 32 = 2 \log 5$; (ii) $3 + \log_{10} 3 = \log_{10} 75 + \log_{10} 40$.

   Why need no base be specified in (i) while it must be in (ii)?
5. Find to what base the logarithms are taken if

   $$2 \log 3 + 3 \log 2 + \log 5 - \log 36 = 0{\cdot}5.$$
6. Find x if $\log x + \log 8 = \log (x + 4) + \log 6$.
7. Without using tables find the value of

   $$\log \tfrac{25}{36} + \log \tfrac{3}{20} - \log \tfrac{5}{48}.$$
8. Establish the following :

   (i) $(1 - \log_b a)(1 + \log_a b) = \log_a b - \log_b a$;

   (ii) $\log_a (a^4 b^4) - \log_b (a^4 b^4) = 4(\log_a b - \log_b a)$.

9. Rewrite each of the following statements without using *log*:

(i) $2\log a=\log b+\log c-\log d$;
(ii) $3\log a=\log d-2\log b$.

If both statements are true with the same values of a, b, c, d, prove $b^7c^3=d^5$.

10. (i) Show that $\log_{16}81=\log_2 3$ and that $\log_{25}10=\frac{1}{2}\log_5 10$.
(ii) If $p=\log_a N$ and $q=\log_{2a} N^2$ find a relation between p, q, a.

Further Examples on Indices and Logarithms

Large and Small Numbers

It is often convenient to use powers of 10 when dealing with either large or small numbers, as for example when concerned with astronomical distances or the weight of an electron; thus

The distance from the earth to the sun is $9{\cdot}3\times10^7$ miles.
The velocity of light is 3×10^{10} cm. per second.
The weight of an electron is 9×10^{-27} gm.

Equations involving Powers

Example I. Solve the equation $3^x=11$.
Here $x\log 3=\log 11$; $\therefore\ x=\log 11\div\log 3=1{\cdot}0414\div{\cdot}4771$.
Using logarithms for the division,

$$x=\text{antilog}\,({\cdot}0174-\bar{1}{\cdot}6786)=\text{antilog}\,({\cdot}3388)=2{\cdot}182.$$

Example II. Find x if $9^x-5\,.\,3^x+6=0$.
If $3^x=z$, then $9^x=z^2$, so that $z^2-5z+6=0$;

$$\therefore\ (z-3)(z-2)=0;\quad \therefore\ z=3 \text{ or } 2.$$
$$3^x=3 \text{ gives } x=1;\quad 3^x=2 \text{ gives } x\log 3=\log 2,$$

from which by division $x=\text{antilog}\,\bar{1}{\cdot}8000={\cdot}631$.
Hence $x=1$ or ${\cdot}631$.

Example III. Solve the simultaneous equations

$$10^x\,.\,4^y=1,\quad 8^x=10^{y+1}.$$

Taking logarithms

$$x+y\log 4=0,\quad x\log 8=y+1,$$
$$\therefore\ -y\log 4\,.\,\log 8=y+1,$$
$$\therefore\ y(1{\cdot}5438)=-1;$$
$$\therefore\ y=-{\cdot}6511.$$

This gives $x={\cdot}3921$

No.	Log.
${\cdot}6021$	$\bar{1}{\cdot}7797$
${\cdot}9031$	$\bar{1}{\cdot}9557$
${\cdot}5438$	$\bar{1}{\cdot}7354$
recip. $1{\cdot}544$	$={\cdot}6511$

Examples 39

[Nos. 1 to 3 large and small numbers.
Nos. 4 to 10 equations with powers.]

1. Find the weight of 10 million million electrons.

2. Given that the diameter of a blood corpuscle is 7×10^{-4} cm., show that the sum of the diameters of the 17×10^{12} blood corpuscles in the body of a man is nearly $1{\cdot}2 \times 10^5$ Km.

3. If light takes 400 years to come from a star at 186,000 miles per second find its distance in miles to 2 significant figures.

Solve the following equations :

4. $2^x = 1{\cdot}5^{12}$. 5. $10^x = {\cdot}6327$. 6. $3 \times 5^{3x} = 5 \times 3^{5x}$.

7. $5^{2x} - 25 \times 5^x + 5 = 0$. 8. $5^{2x^2} - 5^{x^2+1} + 6 = 0$.

9. $2^{2x+y} = 6^y$; $9^x = 3 \times 2^{y+1}$. 10. $\log_{10} x^2y^5 = 4{\cdot}7$; $\log_{10} x^4y^3 = 5{\cdot}2$.

Miscellaneous Examples 40

Express in their simplest form with positive indices :

1. $(8x^{-9})^{-\frac{2}{3}}$. 2. $(27x^6)^{-\frac{2}{3}}$. 3. $(81x^8)^{\frac{3}{4}}$.

4. $3x^{\frac{1}{2}}y^{-\frac{1}{2}} \div \sqrt[3]{(x^{\frac{3}{2}}y^6)}$. 5. $\sqrt[4]{(y^5z^3)} \div \sqrt[4]{(y^{-5}z^{-3})}$.

6. Simplify (i) $8^{-x} \cdot 4^{3x} \cdot 2^x \div 16^{\frac{1}{2}x}$; (ii) $3^a \cdot 9^{2a} \cdot 27^{3a} \div 81^{4a}$.

Work out Nos. 7 and 8.

7. (i) $(a^x - a^{-y})(a^{3x} + a^{2x-y} + a^{x-2y})$;
(ii) $(a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}})$.

8. (i) $(a+b) \div (a^{\frac{1}{3}} + b^{\frac{1}{3}})$; (ii) $(3x^{\frac{1}{2}} + 2y^{\frac{1}{3}} - 4z^{\frac{1}{4}})(3x^{\frac{1}{2}} - 2y^{\frac{1}{3}} + 4z^{\frac{1}{4}})$.

Solve the equations :

9. $4^x = 2^{x-5}$. 10. $3^{4x-7} = (243)^{\frac{1}{4}}$. 11. $12^x = 3^x \cdot 2^{x+7}$.

12. Explain why $\log_2 x = \log_{10} x \div \log_{10} 2$.
Show that $\log_2 x + \log_3 x + \log_4 x \simeq 7{\cdot}079 \log_{10} x$.
[Use tables of logarithms and of reciprocals.]

Without tables give the values of :

13. (i) $\log_2 8$; (ii) $\log_3 81$; (iii) $\log_4 8$;
(iv) $\log_3 \frac{1}{27}$; (v) $\log_{10} {\cdot}001$; (vi) $\log_{0{\cdot}2} 125$.

Use tables to find the values of :

14. $\sqrt[5]{0{\cdot}3845}$. 15. $(0{\cdot}7235)^{-4{\cdot}5}$. 16. $7^{\frac{3}{2}} + 1{\cdot}7^{\frac{2}{3}}$.

17. $(0{\cdot}37)^{0{\cdot}18}$. 18. $\log_8 10$. 19. $\log_2 7{\cdot}146$.

Solve the equations :

20. $15{\cdot}8\, x^{\frac{3}{4}} = (0{\cdot}61)^{\frac{4}{3}}$. 21. $6^{3-2x} \cdot 5^{x+4} = 72$.

22. If $y = 2 \log_{10}(3x+7)$ simplify 10^y.
Solve the equation

$$2 \log_{10}(3x+7) - \log_{10}(2x-1) = 2.$$

D T.A.A.

Exponential Functions

Exponent is an old name for *index*. If in a^x, a is a constant and x is the variable, and we think of a^x as a function of x, we have a new type of function called an *exponential* function.

Different values of a give different exponential functions. The graphs of two such functions, e.g. the graphs of $y=2^x$ and of $y=3^x$ have much in common.

Using the values $2^2=4$, $2^1=2$, $2^0=1$, $2^{-1}=\frac{1}{2}$, $2^{-2}=\frac{1}{4}$, and the corresponding values of 3^x, we obtain a part near the origin for the two graphs shown in Fig. 32.

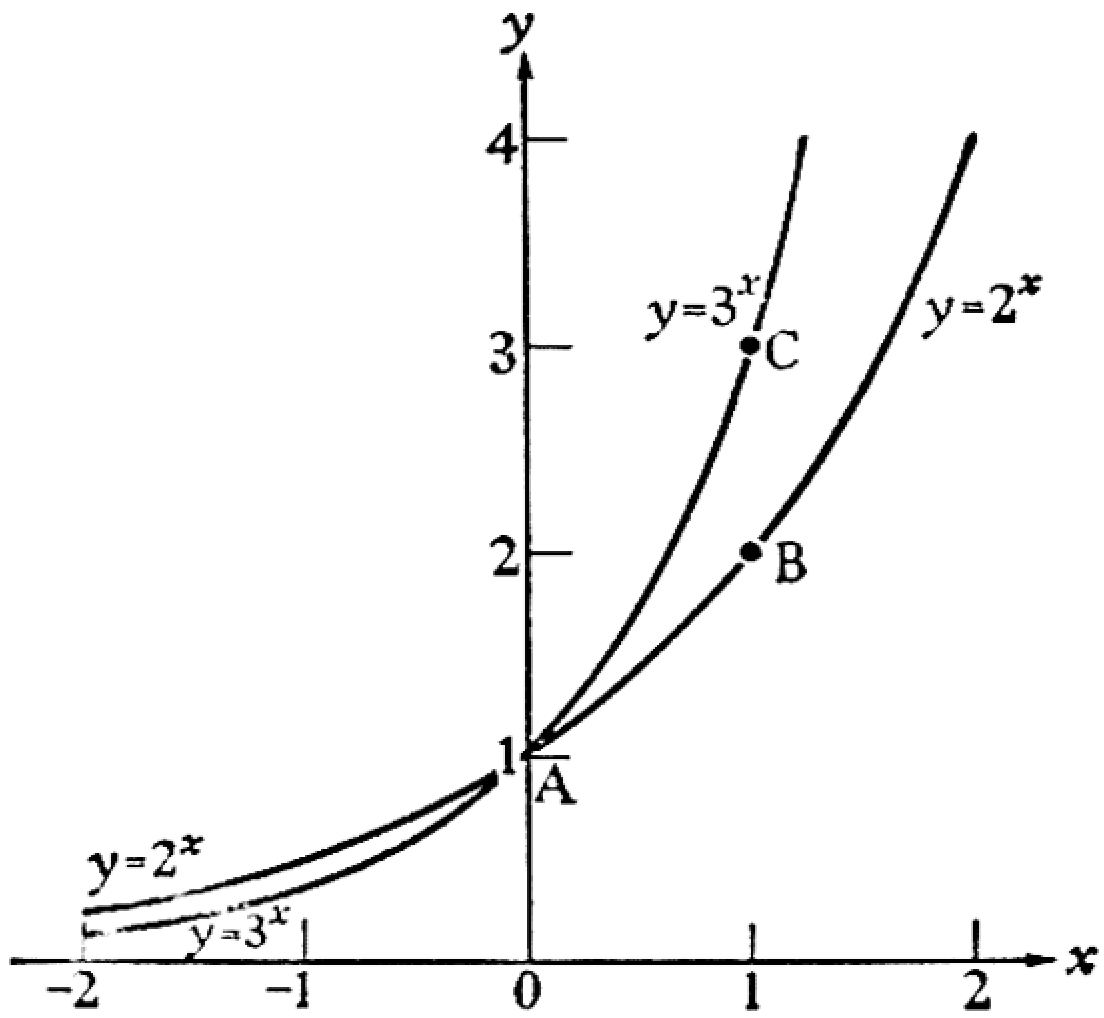

FIG. 32

For all such graphs (if $a>1$, in a^x)

(i) y is always positive;

(ii) y is always increasing as x increases [if $a<1$, y is decreasing];

(iii) y is 1 when $x=0$ and for a^x, y is a when x is 1;

(iv) the value of y for $x=-p$ is the reciprocal of its value for $x=+p$.

Notice that the gradient of $y=2^x$ when $x=0$ is less than 1, which is the gradient of AB in Fig. 32, while the gradient of $y=3^x$ is greater than 1. AC has gradient 2. Thus it is natural to suppose that there is a value of a between 2 and 3 such that the gradient of $y=a^x$ when $x=0$ is exactly 1. This value of a turns out to be a number of great importance in mathematics (see p. 238). It is usually called e and is roughly $2{\cdot}7$.

Examples 41

1. Draw on the same diagram the graphs of $y=2^x$ and $y=(\frac{1}{2})^x$ from $x=-2$ to $x=+2$, and verify that the second graph is the reflexion of the first in the y-axis.
2. On the diagram of No. 1 insert the graph of $y=\frac{1}{2}\{2^x+(\frac{1}{2})^x\}$, which for any value of x comes half-way between the other two graphs.

   [This is a special case of $y=\frac{1}{2}(a^x+a^{-x})$, a curve called the *catenary*, because it is the shape taken by a chain hung up at its two ends.]
3. Draw the graph of $y=1{\cdot}5^x$ from $x=-2$ to $x=3$.
4. Taking 5 in. as the unit on the x-axis and $\frac{1}{2}$ in. as unit on the y-axis, draw the graph of $x=10^x$ from $x=0$ to $x=1$. Put the graph of $y=2^x$ on the same axes and with the same scales.
5. Since $y=2^x$ is the same as $x=\log_2 y$, explain why turning the paper sideways and looking *through* it from the back gives a graph showing logarithms to base 2.

   Use Fig. 32 in this way to see that $\log_2 3 \simeq 1{\cdot}6$ and calculate the value more accurately using tables.
6. Sketch the graphs of

   (i) $y=2^{x-1}$; (ii) $y=3^x-1$; (iii) $y=2^{-x-1}$.
7. Sketch on the same diagram the graphs of $y=2^{x-2}-2$ and $y=3^{-x+2}-2$ from $x=0$ to $x=4$ and find where they meet.

   [Such curves are also called exponential curves, differing from $y=a^x$ only in position and not in character.]
8. Using the same axes as for the graph in Example 6 (i), plot the graph of $y=3-x$, and hence find an approximate solution of the equation $2^{x-1}+x=3$.
9. Using the graph in No. 6 (ii) and the graph of $y=12x-10$ show that there are two roots of the equation $3^x=12x-9$, i.e. of

$$3^{x-1}=4x-3.$$

10. By drawing the graphs of $y=2^x$ and $y=\dfrac{4}{x}$ on the same axes, show that there can be only one root of the equation $x\,.\,2^{x-2}=1$ and find its approximate value.

The Three-ordinate Property

For any function of the type a^x, if x is given the three values $p-q$, p, $p+q$, the corresponding values of y are

$$y_1=a^{p-q},\ y_2=a^p,\ y_3=a^{p+q},$$

and these have the property $y_1y_3=y_2^2$.

This may be expressed by saying that for 3 equally spaced ordinates, the middle one is the mean proportional between the other two or, in other words, is the square root of their product.

[If $h : x = x : k$ we say that x is the mean proportional between h and k, and $x^2 = hk$.]

If $f(x) = a^x$, then $f(p-q) \cdot f(p+q) = \{f(p)\}^2$.

It follows that the x-axis and two ordinates completely determine the curve.

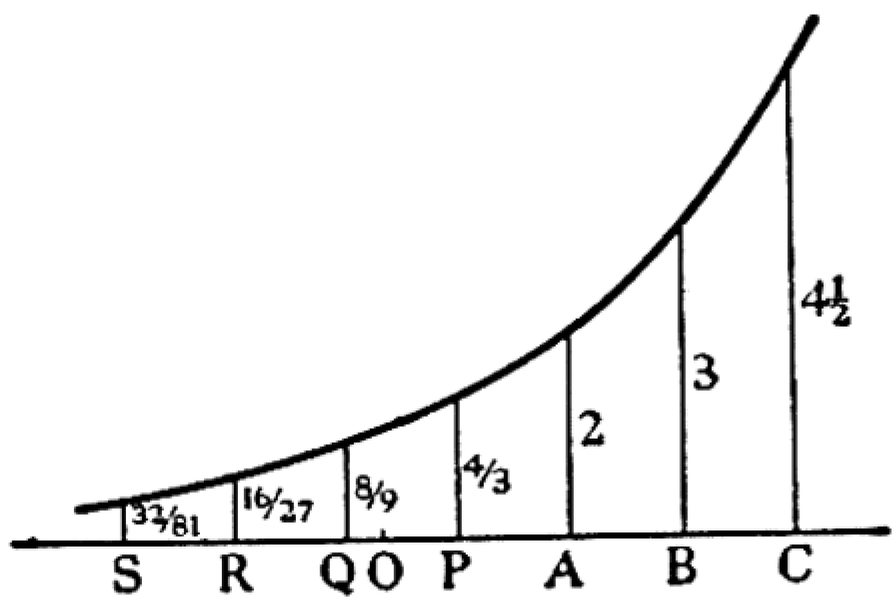

FIG. 33

Consider, for instance, the case where the ordinates at 2 points A and B, 1 cm. apart, are respectively 2 cm. and 3 cm. (Fig. 33). If C is 1 cm. to right of B and the ordinate at C is y_C, then

$$y_C \times 2 = 3^2 ; \text{ i.e. } y_C = 4{\cdot}5.$$

Similarly if P is 1 cm. to left of A and the ordinate is y_P, then

$$y_P \times 3 = 2^2 ; \text{ i.e. } y_P = 4/3.$$

The ordinates at Q, R, S can be found in the same way to be 8/9, 16/27, 32/81, and intermediate ordinates can be filled in if desired; thus the ordinate midway between points A and B is $\sqrt{6}$.

Next consider how to find the equation of the curve.

At the origin the value of y is 1, so the origin is between Q and P and about $\cdot 3$ cm. to the right of Q.

Also if the equation is $y = a^x$ and $OA = p$ we have $a^p = 2$, $a^{p+1} = 3$, whence by division $a = (\frac{3}{2})$, and the equation is $y = (\frac{3}{2})^x$.

Note that such a curve as $y = 2^x - 1$ will not possess the three-ordinate property unless the ordinates are taken from $y = -1$; on the other hand, $y = a^{bx}$ will possess the property.

Examples 42

1. A curve of the type $y=a^{cx-b}$ has ordinates 1 and 4 when $x=2$ and $x=6$ respectively. Use three-ordinate property to find the ordinates for $x=0$, 4, 8 and show that $a=2$, $b=1$, $c=\frac{1}{2}$.

   Measure the ordinates when $x=7$ and compare the measured value with that found by calculation.

2. Where does the curve of No. 1 meet the curve $y=2^{5-x}$?

3. If the ordinates in Fig. 33 are doubled, show that the curve $y=(\frac{3}{2})^x$ passes through the tops of the ordinates, provided that the new origin is between S and R about 0·7 cm. from S.

4. The curve $y=a^x$ is drawn, and points P, Q, R, S are points on the x-axis in that order as x increases and separated by unit intervals; the ordinates at Q and R are respectively $\frac{3}{5}$ and 2 units. Find the value of a and the approximate position of the origin. Also find the ordinates at P and S.

5. On the same axes with 1 inch representing 1 unit on the x-axis and 5 units on the y-axis draw carefully the graphs of $y=2^x$ and $y=x^2+1$ between $x=-1$ and $x=5$. Find the three roots of the equation $2^x=x^2+1$ and give the ranges of value of x for which $2x$ is greater than x^2+1.

Gradient of Exponential Curve

Two points on the curve $y=a^x$ (Fig. 34) are (x, a^x) and $(x+h, a^{x+h})$. The gradient of the chord joining these points is

$$\frac{a^{x+h}-a^x}{h}, \text{ i.e. } a^x \cdot \frac{a^h-1}{h}$$

Of these two factors, the second is *independent of* x, depending only on a and h.

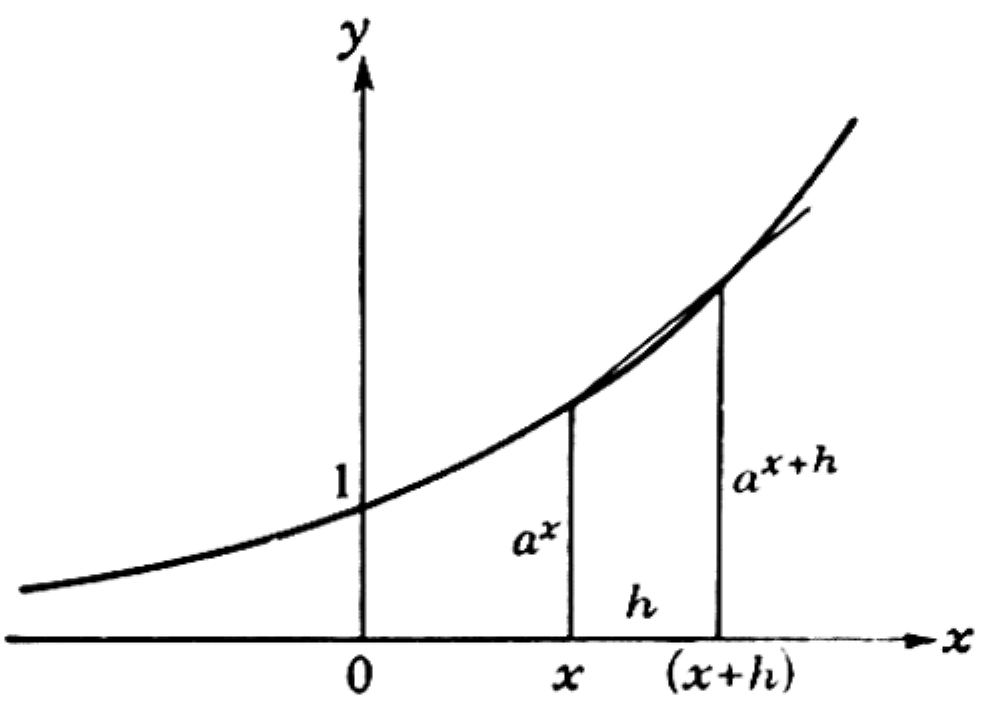

FIG. 34

To obtain the gradient of the tangent at the first of the two points, we suppose h to become small and tend to zero.

Then $\dfrac{a^h - 1}{h}$ approaches a limiting value, and if this value is A, we see that the gradient of the tangent to $y=a^x$ is $a^x \,.\, A$, where A is independent of x and is a function of a only.

Also notice that since $a^0=1$, A is the gradient of $y=a^x$ at the point where it crosses the y-axis, the point (0, 1).

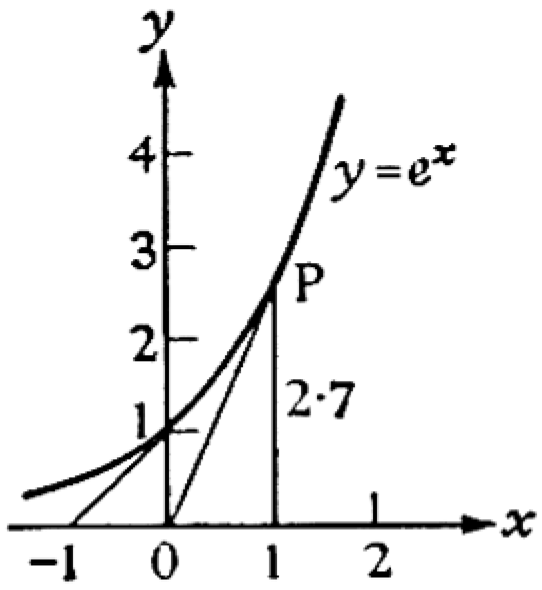

FIG. 35

It has been stated (p. 88) that there is a value of a between 2 and 3 usually called e and approximately 2·7 such that $y=e^x$ has gradient 1 where $x=0$; it follows that $\dfrac{e^h - 1}{h}$ tends to value 1 as h tends to 0, and that if $y=e^x$ the gradient of the curve at any point is e^x or y;—the ordinate and the gradient are given by the same number.

Thus for the curve $y=e^x$, where $x=1$ the value of y is e^1 or e (2·7 approx.); the gradient of the curve is also $\simeq 2·7$.

It follows that the tangent at P (Fig. 35) is the line joining P to the origin.

Using the calculus notation,

$$\text{if } y=e^x, \text{ then } \frac{dy}{dx}=e^x.$$

Those who know the rule for differentiating a "function of a function", namely $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$ when y is a function of u which is a function of x, will see, using $u=kx$, that

$$\text{if } y=e^{kx}, \text{ then } \frac{dy}{dx}=e^{kx}\,.\,k.$$

This result enables us to find the gradient of the general exponential curve $y=a^x$.

For since $a=e^{\log_e a}$, $a^x=(e^{\log_e a})^x=e^{x\log_e a}$, and so if $y=a^x$,

$$\frac{dy}{dx}=e^{x\log_e a}\,.\,\log_e a=a^x\,.\,\log_e a.$$

$\therefore$ the value of the limit $\dfrac{a^h - 1}{h}$ as $h \to 0$, denoted by A above, is $\log_e a$.

Thus in finding gradients to exponential curves, and for other reasons which will be seen later, the process of changing logarithms from base 10 to base e and vice versa is important.

Remember that

$$\mathbf{y = e^x, \quad gradient = e^x}$$
$$\mathbf{y = a^x, \quad gradient = a^x \,.\, \log_e a.}$$

Examples 43

1. Being given that $e = 2{\cdot}718$ find

   (i) $\log_{10} e$; (ii) $\log_e 10$; (iii) $\log_e 2$; (iv) $\log_e 3$; (v) $\log_e 5$.

   [Solution. (i) The tables give $\log_{10} e$ as $\cdot 4343$.

   (ii) $\log_e 10$ is the reciprocal of this, $\simeq 2{\cdot}302$ (it is nearer $2{\cdot}3026$).

   (iii) $\log_e 2 = \log_{10} 2 \times 2{\cdot}3026 = {\cdot}6931$.]

2. (i) Where and at what angle do the curves $y = e^x$ and $y = e^{-x}$ cut?

   (ii) Find the gradients where $x = 0$ and where $x = 1$ for the curves $y = 2^x$ and $y = 3^x$.

3. Show that the tangent to e^x at the point where $x = 0$ passes through the point $(-1, 0)$.

4. Compare the gradients of 2^x where $x = -1$ and of 2^{-x} where $x = 1$.

5. Show that for $y = e^x$, the *subtangent*, i.e. the part of the x-axis between the foot of the ordinate and the point where the tangent cuts it, is always of unit length.

   State the corresponding property for the curve $y = a^x$.

6. Show that the equation $e^x - 3x = 0$ has one root between $x = 0$ and $x = 1$ and another root between $x = 1$ and $x = 2$. Find the smaller root to 2 decimal places.

7. Solve $10^x + x - 2 = 0$ giving the root to two decimal places.

8. Eliminate y from the definitions

   $$2 \cosh y = e^y + e^{-y}, \quad 2 \sinh y = e^y - e^{-y}$$

   to show that $\cosh^2 y - \sinh^2 y = 1$.

9. (i) By considering the graphs of $y = e^x$ and $y = x + a$, show that the equation $e^x - x - a = 0$ has no roots if $a < 1$.

   (ii) Show that the equation $e^x + x - a = 0$ has one root whatever the value of a.

10. P, Q, R are three points on the curve $y = e^x$ with x-coordinates $x - 1$, x, $x + 1$ respectively. The tangent to the curve at Q meets the ordinates of P and R at H and K. Show $HP \,.\, KR = e^{2x}(1 - 2/e)$.

Examples 44 (Miscellaneous)

1. Find the value of $(64 - 64^{\frac{1}{2}} - 64^{\frac{1}{6}}) \div 27^{\frac{2}{3}}$.
2. Find x from (i) $x^{\frac{3}{2}} = 8 \,.\, x^{-\frac{3}{2}}$; (ii) $\log x + \log 2 = 1$.
3. Find an equation connecting x and y if $x = \log_2 z$ and $y = \log_8 z$.
4. Solve the equations (i) $x - 5\sqrt{x} + 6 = 0$; (ii) $x^{\frac{3}{2}} - 5x + 6x^{\frac{1}{2}} = 0$.
5. Use tables to find (i) $\log_3 17$; (ii) $\log(\log_2 12)$.
6. (i) Explain the property corresponding to the three-ordinate property for the curve $y = 3^x + 4$.
   (ii) If $y = a^x + 2$ cuts the curve of (i) where $x = -1$ find the value of a.
7. Calculate $10^{\frac{3}{4}}$ by the square root tables (or the rule for square root); check using logarithms.
8. Without use of tables prove that

   (i) $\log 75 - \log 3 = 2 \log 5$; (ii) $\log\left(\dfrac{\log 256}{\log 2}\right) = 3 \log 2$;

   and express using indices and not logarithms:
   (iii) $\log_5 625 = 4$; (iv) $\log_8 32 = 5/3$; (v) $\log a^{25} = 5 \log a^5$.
9. If light travels at 3×10^{10} cm. per second, find how long it takes to travel 30 km.
10. The function $f(x) \equiv a^x - b$ is such that

    $$f(0) = -2 \text{ and } f(6{\cdot}024) = 0.$$

    Find the values of a and b.
11. (i) If $2 \log_{10} x + 5 \log_{10} y = 1{\cdot}232$ express x in terms of y in a form not involving logarithms.
    (ii) Solve the equation $3^{4x+2} - 5 \times 3^{2x+1} + 4 = 0$. (L.)
12. If $p^2 + q^2 = 14pq$ prove that

    $$\log \frac{p+q}{4} = \tfrac{1}{2}(\log p + \log q).$$

13. Find by using logarithms the value of $\sqrt{(a^2 + b^2)}$ when $a^5 = 0{\cdot}071$ and $b^{0{\cdot}6} = 0{\cdot}248$. (N.U.J.B.)
14. Two numbers x and y are known to be connected by an equation of the form $x^\alpha y^\beta = 10$ where α and β are constants. By experiment it is found that when $x = 2$, $y = 1{\cdot}209$, and when $x = 2{\cdot}5$, $y = 1{\cdot}073$. By taking logarithms, calculate the values of α and β, each correct to two significant figures. (N.U.J.B.)
15. Find (i) a value of n for which $n^9 2^{-n}$ is greater than 10^5, and (ii) a value of n for which it is less than 10^{-5}. (N.U.J.B.)
16. Without using tables, find the value of $(\log_2 3)(\log_3 4)$. (N.U.J.B.)

17. (i) Find, *without using tables*, the value of y when $x=9$ if

$$2 \log_{10} y - 3 \log_{10}\sqrt{x} = 1.$$

(ii) Simplify $\log\left(\dfrac{a-b}{a+b}\right)^2 \div \log\left(\dfrac{a-b}{a+b}\right)$.

(iii) Evaluate, using tables, $\dfrac{13{\cdot}93 - 7{\cdot}27}{13{\cdot}93 \times 7{\cdot}27}$. (L.)

18. With the aid of logarithms, evaluate $(0{\cdot}01579)^{-0{\cdot}4} - (0{\cdot}01579)^{0{\cdot}4}$. (L.)

19. Draw the curve $y=\log_{10} x$ for values of x from $x=\frac{1}{2}$ to $x=5$ taking 1 inch to represent $\frac{1}{10}$ unit on the y-scale and 1 inch to represent 1 unit on the x-scale.

Hence find, graphically, a solution of the equation $8 \log_{10} x = x$. (L.)

20. If $a=b^3$ show that $\log_b x = 3 \log_a x$

Given
$$\log_a (x^2y^3) = z,$$
$$\log_b \left(\frac{x}{y}\right) = w$$
$$\text{and } a = b^3,$$

find $\log_a x$ in terms of z and w. (L.)

21. (*a*) If $a^2+b^2-6ab=0$, prove that

$$2 \log_e (a+b) - \log_e a = \log_e 8b.$$

(*b*) If $\dfrac{x}{a}=(2{\cdot}718)^b$ where $x-a=634$ and $b=0{\cdot}725$, find x. (L.)

22. The population of a town increases at the rate of 8 per cent. per annum. If it is at present 40,824, find by aid of the formula

$$A=P\left(1+\frac{r}{100}\right)^n$$

what it was 2 years ago and also what it will be in 20 years' time if it continues to increase at the same rate. (L.)

23. In a certain town the numbers of births and deaths in each year are respectively $\frac{1}{40}$ and $\frac{1}{48}$ of the population at the beginning of the year. In how many years will the population be doubled, if all other causes of change (such as migration) of the population are neglected? (L.)

24. At the beginning of the year 1909 the population of a certain town was 250,000 while by the end of 1928 it had increased to 312,000. The number of deaths during 1928 was 4,650. Assuming that the numbers of births and of deaths during a year are both proportional to the population at the beginning of the year, and that there is no emigration or immigration, determine the annual birth and death rates per thousand. (L.)

25. The population of England and Wales (in thousands) increased from 32,528 in 1901 to 36,070 in 1911. Supposing the law of increase of population to be expressed by the formula $P=at+b$, where P is the population and t is the time measured in years, and a and b are constants, find (i) the population in 1908, (ii) the year, with decimal fraction from the census-date, when the population was 34 millions. (L.)

26. The number of persons born in any one year is $\frac{1}{45}$ of the population at the commencement of that year, and the number who die in that year is $\frac{1}{60}$ of it. How many years will have elapsed before the population has doubled itself? (L.)

27. (i) Write in descending order of magnitude $2^{-\frac{1}{2}a}$, 2^{-a}, 2^{a} if a is positive. Do this also if a is negative.

(ii) Find the least integer n for which 2^{2n} exceeds 10^{100}. (L.)

28. Prove that $$\frac{x^2+y^2-x^{-2}-y^{-2}}{x^2y^2-x^{-2}y^{-2}}=\frac{xy^{-1}+x^{-1}y}{xy+x^{-1}y^{-1}}.$$ (L.)

29. If $y=\dfrac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}$, express $y+\dfrac{1}{y}$ and $\left(y-\dfrac{1}{y}\right)^2$ in terms of x, in simplest form. (L.)

30. If $a^x=b^y=(ab)^z$, prove that $\dfrac{xy}{x+y}=z$. (L.)

CHAPTER V

POLYNOMIALS ; REMAINDER THEOREM

Polynomials

Algebraic expressions such as

$$2x-5, \quad 3x^2-2ax+7a^2, \quad 4x^2-y^2+3x-7y,$$

have been spoken of as binomials, trinomials and quadrinomials respectively, meaning expressions of two, three and four terms.

Expressions of more than four terms are called " polynomials ", i.e. expressions of many * terms, but this word is also used loosely to include three- and four-termed expressions. It is usually restricted to expressions containing only integral powers of the variables.

Multiplication

When two polynomials are to be multiplied together, each term of the first must be multiplied by each term of the second and these " partial products " added.

Example I. Multiply $2x^4+5x-3x^2-7$ by x^2+2-3x.

The first thing to be done is to arrange each expression in descending powers of x (or alternatively, each in ascending powers). If any power of x is missing it is well to insert it, with zero as coefficient ; so here we get

$$(2x^4+0x^3-3x^2+5x-7)(x^2-3x+2).$$

It will be seen that the product starts with x^6 and contains the other powers of x from x^5 down to x and a constant.

Any power can be picked out ; thus the coefficient of x^3 is

$$0\times 2+(-3)(-3)+5\times 1 \text{ or } 14.$$

It is probably easier to arrange the work as below, something like long multiplication in arithmetic :

$$\begin{array}{rrrrrrr}
 & & 2x^4 & +0x^3 & -3x^2 & +5x & -7 \\
 & & & & x^2 & -3x & +2 \\
\hline
2x^6 & +0x^5 & -3x^4 & +5x^3 & -7x^2 & & \\
 & -6x^5 & -0x^4 & +9x^3 & -15x^2 & +21x & \\
 & & 4x^4 & +0x^3 & -6x^2 & +10x & -14 \\
\hline
2x^6 & -6x^5 & +x^4 & +14x^3 & -28x^2 & +31x & -14
\end{array} \quad \text{Answer.}$$

* It is never easy to decide the smallest number to be called " many ". Compare the phrase " two's company, three's a crowd ".

In such cases it saves time and actually increases accuracy to *detach the coefficients* and arrange the work thus

$$\begin{array}{r}
2+0-3+\ 5-\ 7 \qquad\qquad\\
1-3+\ 2 \qquad\qquad\\
\hline
2+0-3+\ 5-\ 7 \qquad\qquad\\
-6-0+\ 9-15+21 \qquad\\
4+\ 0-\ 6+10-14\\
\hline
2-6+1+14-28+31-14
\end{array}$$

Inserting the powers of x from x^6 downwards we get the answer.

This work with detached coefficients will apply equally well if the question is " Find the product of $2x^4+5xy^3-3x^2y^2-7y^4$ by $x^2-3xy+2y^2$, in which each expression is *homogeneous* (i.e. composed of terms of the same degree) in two letters.

Example II. Multiply $2x^2+3x+1$ by x^2+2x+1 and compare with the work of multiplying 231 by 121.

Solution :

Using detached coefficients for the algebra :

$$\begin{array}{r}
2+3+1 \qquad\qquad\\
1+2+1 \qquad\qquad\\
\hline
2+3+1 \qquad\qquad\\
4+6+2 \qquad\\
2+3+1\\
\hline
2+7+9+5+1
\end{array}$$

Here the arithmetic is almost identical :

$$\begin{array}{r}
231\quad\\
121\quad\\
\hline
231\quad\\
462\ \ \\
231\\
\hline
27951
\end{array}$$ Answer.

Answer : $2x^4+7x^3+9x^2+5x+1$.

The exact correspondence depends on there being no carrying in the arithmetic. Had the multiplier been 123, the coefficients in the algebra would have been 2, 7, 13, 11, 3, while the arithmetic would have given 28413 as the answer, carrying being needed in two places.]

Example III. Multiply $x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+xa^{n-2}+a^{n-1}$ by $x-a$.

In this important example some imagination has to be used to picture terms which are not written down. The product, like the expressions to be multiplied, will be homogeneous.

Solution :

$$\begin{array}{llll}
x^{n-1}+ax^{n-2}+a^2x^{n-3}\ldots & & +xa^{n-2}+a^{n-1} & \\
\qquad\qquad x-a & & & \\
\hline
x^n+ax^{n-1}+a^2x^{n-2}+\ldots & & & +x^2a^{n-2}+xa^{n-1}\\
\quad -ax^{n-1}-a^2x^{n-2}-\ldots & & & -x^2a^{n-2}-xa^{n-1}-a^n\\
\hline
\end{array}$$

$$x^n-a^n \quad \text{Answer.}$$

Note carefully the result, that for all values of n

$$\mathbf{x^n - a^n \text{ has } x - a \text{ as a factor.}}$$

This is used in proving the very important *Remainder Theorem*, p. 105.

Examples 45

1. Multiply (i) x^4+5x^3-3x+2 by x^4+5x^2+3x+2.
[Detach coefficients, if it seems desirable to do so.]

(ii) x^3+7x^2-1 by $2x^5+4x^4+3x^3+x+3$.

2. Compare the multiplication of x^2+7x+1 by x^2+x+1 with that of 171 by 111.

Also $(x^3+2x+1)\times(x^2+1)$ with 1021×101.

3. Multiply $x^{2n}-ax^{2n-1}+a^2x^{2n-2}-\ldots-a^{2n-1}x+a^{2n}$ (in which the signs alternate) by $x+a$ and also by $x-a$.

4. Use detached coefficients to work out:
(i) $(1+x)^3\times(1+x)^2$; (ii) $(1+x+x^2)^3$; (iii) $(a^2+2ab+3b^2)^3$.

5. Prove that $x^2+(x+1)^2+\{x(x+1)\}^2\equiv(x^2+x+1)^2$
[e.g. $6^2+7^2+42^2=43^2$], and express 11^2+12^2 as the difference of two squares.

6. Multiply (i) $3x+6x^{\frac{1}{2}}+4$ by $2x+x^{\frac{1}{2}}+8$;
(ii) $x-2x^{\frac{2}{3}}+x^{\frac{1}{3}}-1$ by $2x+x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$.

Binomial Expansions

A specially important use of detached coefficients is in determining the coefficients in successive powers of $1+x$, or of $a+b$. Since the multiplier is always $1+1$, it can be remembered and not written down; the work is as follows.

$$\begin{array}{llllll}
1+1 & & & & & \\
\quad 1+ & 1 & & & & \\
\hline
1+2+ & 1 & & & & \text{square} \\
\quad 1+ & 2+ & 1 & & & \\
\hline
1+3+ & 3+ & 1 & & & \text{cube} \\
\quad 1+ & 3+ & 3+1 & & & \\
\hline
1+4+ & 6+ & 4+1 & & & \text{fourth power} \\
\quad 1+ & 4+ & 6+4+1 & & & \\
\hline
1+5+ & 10+ & 10+5+1 & & & \text{fifth power}
\end{array}$$

Thus $(1+x)^5=1+5x+10x^2+10x^3+5x^4+x^5$
and $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

The work can be shortened, so that only the successive answers are written down, by the use of the rule, which can be seen from the above work, that " each coefficient is the sum of that above it and that next to the left above it ".

$$\begin{array}{l}
1+1\\
1+2+\ 1\\
1+3+\ 3+\ 1\\
1+4+\ 6+\ 4+\ 1\\
1+5+10+10+\ 5+\ 1\\
1+6+15+20+15+\ 6+1\\
1+7+21+35+35+21+7+1
\end{array}$$

This triangle of figures, which can be continued as far as desired, is known as "Pascal's Triangle ".

These figures are used in finding powers of any binomial, e.g. for $(1+2x)^7$ the successive coefficients are

$$1,\ 7\times 2,\ 21\times 2^2,\ 35\times 2^3,\ 35\times 2^4,\ 21\times 2^5,\ 7\times 2^6 \text{ and } 2^7,$$

or again

$$(3a-2b)^5 = 3^5a^5 - 3^4 \cdot 2 \cdot 5a^4b + 3^3 \cdot 2^2\ 10a^3b^2 \\ - 3^2 \cdot 2^3 \cdot 10a^2b^3 + 3 \cdot 2^4 \cdot 5ab^4 - 2^5b^5,$$

all odd powers of b having a negative coefficient, and the expression being homogeneous.

A rule by which the coefficients of any power can be found without reference to those of the previous powers is discussed in Chapter VIII.

Examples 46

1. Carry on the above work as far as $(1+x)^{12}$ and find the coefficient of x^2 in $(1+x)^{13}$.

2. Examine the vertical columns of figures in the triangle above and explain how each coefficient can be found as the sum of figures in the previous column.

3. Examine the sums of successive rows of figures in the triangle. Explain the result.

4. Write out in full $(3x+y)^4$; $(1-2x)^5$; $(2a+3b)^6$.

5. Write $(1+x\sqrt{3})^7$ in the form $P+Q\sqrt{3}$ when P and Q do not contain surds. If $x=1$ show that $P-Q=240$.

6. If $(1+2\sqrt{2})^6$ is written in the form $P+Q\sqrt{2}$ where P and Q do not contain surds, show that $Q=1100$.

7. Write in full $(1-x)^7$ and find the result if this is added to $(1+x)^7$.

8. Simplify $(1+x)^6-2(1+x)^3(1-x)^3+(1-x)^6$.

Division

Since division is the reverse process to multiplication, it is possible to assume a quotient and remainder and then compare coefficients.

Example I. Divide $2x^3-4x^2+6x-5$ by $x-3$.

The quotient will be of the second degree and the remainder a constant, so assume $2x^3-4x^2+6x-5=(ax^2+bx+c)(x-3)+d$.

Equating coefficients of x^3, $a=2$;

$$\begin{array}{lll} x^2, & -3a+b=-4, & \therefore\ b=2; \\ x, & -3b+c=6, & \therefore\ c=12; \\ \text{constants} & -3c+d=-5, & \therefore\ d=31. \end{array}$$

It is, however, easier to arrange the work like long division in arithmetic:

$$\begin{array}{r|l|l} x-3 & 2x^3-4x^2+6x-5 & 2x^2+2x+12 \quad \text{Quotient} \\ & 2x^3-6x^2 & \\ & \quad 2x^2+6x & \\ & \quad 2x^2-6x & \\ & \qquad 12x-5 & \\ & \qquad 12x-36 & \\ & \qquad\quad 31 \quad \text{Remainder.} & \end{array}$$

Example II. Divide $x^4+4x^3+7x^2-13x-65$ by x^2-x-6.

[Coefficients may be detached as in multiplication.]

Using detached coefficients:

$$\begin{array}{r|l|l} 1-1-6 & 1+4+7-13-65 & 1+5+18 \\ & 1-1-6 & \\ & \quad 5+13-13 & \\ & \quad 5-5-30 & \text{Quotient} \quad x^2+5x+18 \\ & \qquad 18+17-65 & \\ & \qquad 18-18-108 & \text{Remainder } 35x+43 \\ & \qquad\quad 35+43 & \end{array}$$

The remainder is normally of one degree lower than the divisor; in Example I it is a constant, in Example II it is linear.

Example II may also be done using the factors $x-3$, $x+2$:

$$
\begin{array}{r|l|l}
\underline{1-3} & 1+4+\ 7-13-\ 65 & \underline{1+7+28+71} \\
 & \underline{1-3} & \\
 & \qquad 7+\ 7 & \\
 & \qquad \underline{7-21} & \\
 & \qquad\quad 28-13 & \\
 & \qquad\quad \underline{28-84} & \\
 & \qquad\qquad 71-\ 65 & \\
 & \qquad\qquad \underline{71-213} & \\
 & \qquad\qquad\qquad 148 &
\end{array}
$$

$$
\begin{array}{r|l|l}
\underline{1+2} & 1+7+28+71 & \underline{1+5+18} \\
 & \underline{1+2} & \\
 & \quad 5+28 & \\
 & \quad \underline{5+10} & \\
 & \qquad 18+71 & \\
 & \qquad \underline{18+36} & \\
 & \qquad\qquad 35 &
\end{array}
$$

Quotient $x^2+5x+18$; Remainder $35(x-3)+148=35x+43$.

Example III. Divide $x^{2n}+a^{2n}$ by $x+a$.

[Terms not written down must be pictured.]

$$
\begin{array}{r|l|l}
\underline{x+a} & x^{2n}+0 \qquad +0 \qquad +0+\dots+a^{2n} & \underline{x^{2n-1}-ax^{2n-2}+a^2x^{2n-3}\dots} \\
 & \underline{x^{2n}+ax^{2n-1}} & \\
 & \quad -ax^{2n-1}+0 & \\
 & \quad \underline{-ax^{2n-1}-a^2x^{2n-2}} & \\
 & \qquad\qquad a^2x^{2n-2}\ \ +0 &
\end{array}
$$

The terms in the quotient will have numerical coefficients $+1$ and -1 alternately, and as there will be $2n$ of them the last coefficient will be -1.

Thus the last steps will be:

$$
\begin{array}{l|l}
a^{2n-2}x^2+0 \qquad +a^{2n} & \underline{\dots+a^{2n-2}x-a^{2n-1}} \\
\underline{a^{2n-2}x^2+a^{2n-1}x} & \\
\qquad -a^{2n-1}x+a^{2n} & \\
\qquad \underline{-a^{2n-1}x-a^{2n}} & \\
\qquad\qquad\quad 2a^{2n} &
\end{array}
$$

$\therefore$ the quotient is $x^{2n-1}-ax^{2n-2}+a^2x^{2n-3}-\dots+a^{2n-2}x-a^{2n-1}$, and the remainder is $2a^{2n}$.

Fractional coefficients are liable to occur.

Example IV. Divide $5x^3 - 4x + 2$ by $7x + 3$.

$$
\begin{array}{r|l}
 & \frac{5}{7}x^2 - \frac{15}{7^2}x - \frac{151}{7^3} \\
\hline
7x+3 & 5x^3 + 0 - 4x + 2 \\
 & 5x^3 + \frac{15}{7}x^2 \\
\hline
 & \quad -\frac{15}{7}x^2 - 4x \\
 & \quad -\frac{15}{7}x^2 - \frac{45}{7^2}x \\
\hline
 & \qquad\qquad -\frac{151}{7^2}x + 2 \\
 & \qquad\qquad -\frac{151}{7^2}x - \frac{453}{7^3} \\
\hline
 & \qquad\qquad\qquad\quad \frac{1139}{7^3}
\end{array}
$$

Quotient $\frac{5}{7}x^2 - \frac{15}{7^2}x - \frac{151}{7^3}$

Remainder $\frac{1139}{7^3}$

Example V. Division may also be worked in ascending powers of x, and this is usually employed when x is small.

Divide $2 - 4x + 5x^3$ by $3 + 7x$.

$$
\begin{array}{r|l}
 & \frac{2}{3} - \frac{26}{3^2}x + \frac{182}{3^3}x^2 \\
\hline
3+7x & 2 - 4x + 0 + 5x^3 \\
 & 2 + \frac{14x}{3} \\
\hline
 & \quad -\frac{26x}{3} + 0 \\
 & \quad -\frac{26x}{3} - \frac{182}{3^2}x^2 \\
\hline
 & \qquad\qquad \frac{182}{3^2}x^2 + 5x^3 \\
 & \qquad\qquad \frac{182}{3^2}x^2 + \frac{1274}{3^3}x^3 \\
\hline
 & \qquad\qquad\qquad -\frac{1139}{3^3}x^3
\end{array}
$$

Quotient $\frac{2}{3} - \frac{26}{9}x + \frac{182}{27}x^2$

Remainder $-\frac{1139}{27}x^3$

Notice that both quotient and remainder are completely different from those in Example IV, when the same division was worked in descending

powers. Using ascending powers the division may be continued indefinitely as follows :

$$
\begin{array}{r|l}
 & -\dfrac{1139}{3^4}x^3 + 1139 \cdot \dfrac{7}{3^5}x^4 \\ \hline
3+7x & -\dfrac{1139}{3^3}x^3 + 0 \\
 & -\dfrac{1139}{3^3}x^3 - \dfrac{1139}{3^4} \cdot 7x^4 \\ \cline{2-2}
 & \qquad\qquad \dfrac{1139}{3^4} \cdot 7x^4
\end{array}
$$

The quotient will continue with

$$- 1139\left\{\frac{7^2}{3^6}x^5 - \frac{7^3}{3^7}x^6 + \frac{7^4}{3^8}x^7 - \ldots\right\}$$

At whatever stage the division is stopped, there is a remainder. (See Example 47, Nos. 5, 6.)

Examples 47

1. Divide $3x^3 - 2x^2 + x + 7$ by $x^2 - 2x + 5$.
2. Divide $a^3 + 7a^2b + 5ab^2 - 4b^3$ by $2a + 3b$ in descending powers of a, showing that the remainder is $\frac{7}{8}b^3$.
3. Divide (i) $x^3 + a^3$, (ii) $x^6 + a^6$ by $x - a$.
4. If the division of No. 1 is worked in ascending powers show that the first two figures in the quotient are $\dfrac{7}{5} + \dfrac{19x}{25}$ and state the remainder at this stage.
5. Carry out the division of Example 2 in descending powers of b, showing that in this case the remainder is $-7a^3/27$.
6. Find a series for [illegible] $+ 2x + x^2)$ in ascending powers of x by division. What is the remainder when the term in x^8 has been reached in the quotient?
7. Divide $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$ by $a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.

   [It might save trouble to write p, q for $a^{\frac{1}{4}}$, $b^{\frac{1}{4}}$.]
8. (*a*) Divide 1 by $1 - x$ in ascending powers of x as far as the term in $x^{[illegible]}$, and give the remainder.

   (*b*) As in (*a*), but dividing 1 by $(1 - x)^2$.
9. If the division of $f(x)$ by $\phi(x)$ gives a quotient $Q(x)$ and a remainder $R(x)$, show that any common factor of $f(x)$ and $\phi(x)$ is also a factor of $R(x)$.

[This result is the basis of the "long rule" for finding Highest Common Factor (H.C.F.). The H.C.F. of $f(x)$ and $\phi(x)$ is also that of $\phi(x)$ and $R(x)$. We start again with these and continue the process till a stage is reached when the division is exact. The division is then the H.C.F. of the original functions.]

10. Show that when the first of the expressions x^3+2x^2-2x-1 and x^3+x^2-5x-2 is divided by the second the quotient is 1 and the remainder is the common factor of the expressions.

11. If $x^4-3x^3+2x^2-3x+1$ is divided by x^3-3x^2+x-3 show that the remainder after the first step of the division is the H.C.F. of the two expressions.

12. Divide $6x^{\frac{4}{3}}+7x-64x^{\frac{2}{3}}+23x^{\frac{1}{3}}+28$ by $x^{\frac{2}{3}}+3x^{\frac{1}{3}}-4$.

[In such an example put $x^{\frac{1}{3}}=y$, or else detach coefficients.]

The Factor Theorem and Remainder Theorem

If $\qquad f(x)\equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n,$

then $\qquad f(a)=p_0a^n+p_1a^{n-1}+p_2a^{n-2}+\ldots+p_{n-1}a+p_n.$

$$\therefore\ f(x)-f(a)=p_0(x^n-a^n)+p_1(x^{n-1}-a^{n-1})+p_2(x^{n-2}-a^{n-1})$$
$$+\ldots+p_{n-1}(x-a).$$

Now it has been shown (p. 98) that for all values of n, x^n-a^n has $(x-a)$ as a factor,

$\therefore\ f(x)-f(a)$ consists of a series of brackets each having $(x-a)$ as a factor;

$\therefore\ f(x)-f(a)$ has $(x-a)$ as a factor.

From this statement two most important results follow:

The Factor Theorem.

If f(a) = 0, then (x − a) is a factor of f(x). (I)

The Remainder Theorem.

If f(x) is divided by (x − a), the remainder is f(a). (II)

Division by a Binomial

If a polynomial $f(x)$ is divided by $(x-a)$ giving the quotient $Q(x)$ and the remainder R, then R is a constant independent of x and the result of the division may be written in either of two ways:

$$f(x)=(x-a)\,Q(x)+R \quad \ldots\ldots\ldots\ldots\text{(III)}$$

or $$\frac{f(x)}{x-a}=Q(x)+\frac{R}{x-a}\cdot \quad \ldots\ldots\ldots\ldots\text{(IV)}$$

To the student these two statements may appear to be the same, but there is one important distinction between them. (III) is an identity in the sense that if the expression on the right is arranged as a polynomial in x, the coefficients of this polynomial will be exactly the same as those of $f(x)$. Thus the two sides are the *same function and will give equal values for every possible value of* x.

This statement about equal values is also true of (IV) with a single exception. It is not possible to put $x=a$ in (IV), division by 0 being meaningless and not permitted in mathematics.*

As we wish to put $x=a$, the question arises: which of the two statements is proved by the work of division? An example will show this.

Consider the work of dividing $2x^3-4x^2+6x-5$ by $x-3$.

$$\begin{array}{r|l|l}
x-3 & 2x^3-4x^2+6x-5 & 2x^2+2x+12 \\
 & \underline{2x^3-6x^2} & \\
 & \quad 2x^2+6x-5 & \\
 & \quad \underline{2x^2-6x} & \\
 & \qquad 12x-5 & \\
 & \qquad \underline{12x-36} & \\
 & \qquad\qquad 31 &
\end{array}$$

Here $x-3$ has been multiplied in turn by $2x^2$, $2x$ and 12, and the results subtracted from $2x^3-4x^2+6x-5$, leaving finally 31.

(The -5 in the 3rd line is usually understood, but not written down.)

Thus what has been proved is that

$$2x^3-4x^2+6x-5-(x-3)(2x^2+2x+12)=31,$$

i.e. that
$$2x^3-4x^2+6x-5=(x-3)(2x^2+2x+12)+31,$$

which, for this example, corresponds to statement (III) above.

The statement corresponding to (IV) follows.

General case

Statement (III) is proved to be an identity by division.

In this put $x=a$; the result is $f(a)=0\times Q(a)+R$ (R being a constant is the same whatever value is given to x);

$$\therefore\ R=f(a).$$

This is the Remainder Theorem as before.

* Or else from $0\times 3=0\times 5$ one could deduce $3=5$.

The Factor Theorem also follows, being the special case when $f(a)=0$.

Note that the question beginning " Use the Remainder Theorem to factorise . . . " implies the use of the special case.

Factors of Polynomial

Suppose that $f(x)$ is a polynomial of degree n, which is known to be zero for the n values $\alpha_1, \alpha_2, \ldots \alpha_n$ of x.

Then making use of (I), $(x-\alpha_1)$, $(x-\alpha_2)$ ... are factors of $f(x)$ and $f(x)=p_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$ where p_0x^n is the term of degree n in $f(x)$, p_0 being a non-zero constant, independent of x.

Now if x is given any value whatever, except one of the α's, $f(x)$ cannot be equal to zero, for none of the n factors can vanish.

$\therefore$ if f(x) is a polynomial of the nth degree, it cannot vanish for *more* than n values of x unless it is identically zero.

It can only be identically zero if each of the coefficients $p_0, p_1, p_2 \ldots$ is zero, as for example in the case of

$$2x^3-4x^2+6x-36-(x-3)(2x^2+2x+12).$$

It is quite possible that there may *not* be n values of x for which $f(x)$ vanishes *—for example, the cubic $(x-p)(x^2+q^2)$ vanishes only if $x=p$; but what has been shown is that it cannot vanish for *more* than 3 values if it is a cubic.

Example I. Prove that $x+1$ is a factor of $x^3-2x^2-13x-10$ and find the other factors.

Substitute -1 for x in the expression ; the result is $-1-2+13-10=0$.

$\therefore$ $x+1$ is a factor ; $\therefore$ $x^3-2x^2-13x-10=(x+1)(x^2+ax-10)$, the first and last coefficients being seen at once.

By coefficients of x^2 ; $a+1 \;=-2$; $\therefore$ $a=-3$.

Check coefficients of x ; $a-10=-13$.

The factors of $x^2-3x-10$ are $(x+2)(x-5)$.

$\therefore$ the factors of the cubic are $(x+1)(x+2)(x-5)$.

Example II. If $f(x)=2x^3+x^2-15x-18$ show that $f(3)=0$ and find the factors of $f(x)$.

$f(3)=54+9-45-18=0$; $\therefore$ $x-3$ is a factor.

$f(x)=(x-3)(2x^2+ax+6)$ and coefficients of x^2 give $a-6=1$.

The factors of $2x^2+7x+6$ are seen to be $(2x+3)(x+2)$.

$$\therefore\; f(x)=(2x+3)(x+2)(x-3).$$

* In real algebra. If x may be complex (see Chap. XI) there are always n values.

Example III. Suppose that $f(x)$ is divided by $(x-a)(x-b)$ and that the remainder, which is of the first degree, is written in the form

$$A(x-a)+B(x-b).$$

It is required to find A and B.

Solution. Suppose the quotient to be $Q(x)$.

Then by hypothesis

$$f(x)=(x-a)(x-b)\,Q(x)+A(x-a)+B(x-b). \quad \text{.............(I)}$$

[As in the proof of the Remainder Theorem, this can be shown to be true for *all* values of x.]

In (I) put $x=a$; then $f(a)=0+0+B(a-b)$,

In (I) put $x=b$; then $f(b)=0+A(b-a)+0$.

$$\therefore\ A=\frac{f(b)}{b-a} \quad \text{and} \quad B=\frac{f(a)}{a-b}.$$

Examples 48

1. Prove $x-1$ is a factor of x^3+4x^2-x-4 and find the other factors.
2. Prove that both $x-2$ and $x+3$ are factors of $x^4+5x^3-2x^2-24x$ and write the expression in factors.
3. Factorise $x^3-2x^2-11x-6$ and $x^3+5x^2-4x-20$.
4. Factorise (i) $x^4-4x^3-13x^2+4x+12$,
   (ii) $x^4-9x^3+13x^2+9x-14$.
5. Solve the following equations by factorising:
   (i) $x^3-6x^2+5x+12=0$; (ii) $x^3+3x^2-13x-15=0$;
   (iii) $x^6-6x^4+5x^2+12=0$; (iv) $x^4-6x^3+3x^2+10x=0$.
6. Find the remainder when $x^4-2x^3-2x^2-2x+a$ is divided by $x-3$. If a is chosen so that $x-3$ is a factor, find the other factors.
7. Find the value of p for which $x-p$ is a factor of
   $$x^3+px^2-(2p^2+12)x+7p+10,$$
   and for this value of p factorise the expression.
8. What are the remainders for
   (i) $(x^7+2x^2+1)\div(x-2)$? (ii) $(x^5+5x^4-x-3)\div(x+2)$?
9. If the remainder when $f(x)$ is divided by $(x-a)(x-b)$ is $Px+Q$ show that $(a-b)P=f(a)-f(b)$ and find the value of $(a-b)Q$.

Coefficients and Roots

If
$$a_0x^2+a_1x+a_2\equiv a_0(x-\alpha)(x-\beta),$$
and since the R.H.S. is $a_0\{x^2-(\alpha+\beta)x+\alpha\beta\}$, it follows that

$$\alpha+\beta=-a_1/a_0 \quad \text{and} \quad \alpha\beta=a_2/a_0.$$

Again, if $a_1x^3 + a_4x^2 + a_2x + a_3 \equiv a_0(x-\alpha)(x-\beta)(x-\gamma)$
which $\qquad = a_0\{x^3 - (\alpha+\beta+\gamma)x^2 + (\beta\gamma+\gamma\alpha+\alpha\beta)x - \alpha\beta\gamma\}$,
it follows that $\alpha+\beta+\gamma = -a_1/a_0$, $\beta\gamma+\gamma\alpha+\alpha\beta = a_2/a_0$, $\alpha\beta\gamma = -a_3/a_0$,
or $\qquad \Sigma\alpha = -a_1/a_0, \quad \Sigma\beta\gamma = a_2/a_0, \quad \alpha\beta\gamma = -a_3/a_0.$

The same process applies to the general case.

If $\qquad f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$
and if the n factors of the R.H.S. are $(x-\alpha_1)$, $(x-\alpha_2)$, ... $(x-\alpha_n)$, then the R.H.S. $= a_1\{x^n - (\Sigma\alpha_1)x^{n-1} + (\Sigma\alpha_1\alpha_2)x^{n-2} - (\Sigma\alpha_1\alpha_2\alpha_3)x^{n-3} + \ldots\}$.

It follows that $\Sigma\alpha_1 = -a_1/a_0$, $\Sigma\alpha_1\alpha_2 = a_2/a_0$, $\Sigma\alpha_1\alpha_2\alpha_3 = -a_3/a_0$, and so on.

These expressions $\Sigma\alpha_1$, $\Sigma\alpha_1\alpha_2$, $\Sigma\alpha_1\alpha_2\alpha_3$, ... are called *the elementary symmetric functions of* $\alpha_1, \alpha_2, \alpha_3 \ldots$.

It should be understood that, for instance, the term in x^{n-2} in the expansion of $(x-\alpha_2)(x-\alpha_2) \ldots (x-\alpha_n)$ is obtained by taking the x from $(n-2)$ brackets and $(-\alpha)$ from the remaining 2 brackets in every possible way and multiplying them together; again taking x from $(n-3)$ brackets and $(-\alpha)$ from the remaining 3 brackets and multiplying them together gives a term in x^{n-3}, and the sum of such terms is $-(\Sigma\alpha_1\alpha_2\alpha_3)x^{n-3}$, and so on.

As $\alpha_1, \alpha_2, \alpha_3 \ldots$ are the roots of the equation $f(x) = 0$, the relations $\Sigma\alpha_1 = -a_1/a_0$, $\Sigma\alpha_1\alpha_2 = a_2/a_0$, etc., are usually referred to as "the relations between the coefficients and the roots".*

Examples 49

1. What is the sum of the roots for the following equations?

(i) $24x^2 - 48x + 5 = 0$; $\qquad$ (ii) $13x^2 + 2x - 1 = 0$;
(iii) $2x^3 - 3x^2 - 30x + 56 = 0$.

2. If α, β, γ are the roots of the Equation No. 1 (iii), what are the values of (i) $\alpha\beta + \beta\gamma + \gamma\alpha$? (ii) $\alpha\beta\gamma$? (iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$?

3. Use the identity $(\Sigma\alpha_1)^2 \equiv \Sigma\alpha_1^2 + 2\Sigma\alpha_1\alpha_2$ to find the sum of the squares of the roots of the equations:

(i) $x^3 - 2x^2 - 5x + 6 = 0$; $\qquad$ (ii) $x^3 + 2x^2 - 5x - 6 = 0$;
(iii) $2x^3 - 25x^2 + 92x - 105 = 0$; $\qquad$ (iv) $3x^3 + 7x^2 - 2x + 6 = 0$.

4. If $x = \alpha_1$ is a root of No. 3 (i), show that $z = 1/\alpha_1$ satisfies

$$6z^3 + 5z^2 - 2z + 1 = 0.$$

Hence find the sum of the reciprocals of the roots of No. 3 (i).

Similarly find the sum of the reciprocals of the roots of the other equations in No. 3.

* It will be shown in Chap. XI on Complex Numbers that these relations hold good whether there are n real roots or not.

5. If $x=\alpha_1$ is a root of No. 3 (i) show that $y=k\alpha_1$ is a root of

$$y^3-2\,.\,ky^2+5\,.\,k^2y+6\,.\,k^3=0.$$

Form the equation

(i) whose roots are twice those of No. 3 (ii),

(ii) whose roots are half those of No. 3 (iv).

6. If $x^3+px+q=0$ has roots α, β, γ, write down the values of $\alpha+\beta+\gamma$, $\alpha\beta+\beta\gamma+\gamma\alpha$, $\alpha\beta\gamma$.

Also show that $\alpha^3+\beta^3+\gamma^3=-3q$.

[Hint : $\alpha^3=-p\alpha-q$, etc., or use factors of $a^3+b^3+c^3-3abc$.]

7. If $x^3+px+q=0$ has 2 equal roots show $4p^3+27q^2=0$.

[Hint : Let the roots be α, α, β ; write down the relations between the coefficients and the roots and eliminate α and β.]

8. Show that the equation $2(y-2)^3-3(y-2)^2-30(y-2)+56=0$, i.e. $2y^3-15y^2+6y+88=0$ has roots which are each two more than those of No. 1 (iii).

Similarly form the equation whose roots are one less than those of No. 3 (i).

9. Express in terms of the elementary symmetric functions of a_1, a_2, a_3, ... a_n the functions (i) $\Sigma a_1{}^3$, (ii) $\Sigma a_1{}^2 a_2{}^2$.

10. If α, β, γ are the roots of $x^3+px+q=0$ find the values of

$$\text{(i) } \alpha^{-2}+\beta^{-2}+\gamma^{-2}\text{ ; (ii) } \alpha^{-3}+\beta^{-3}+\gamma^{-3}.$$

Expressions involving more than Two Letters

Expressions involving three letters often occur in connection with triangles or with the three coordinates needed to specify position in solid geometry. In multiplying two such expressions it is important to decide beforehand the number and type of terms to be expected, and where possible, to make use of symmetry. The expressions will usually be homogeneous in the three letters.

Example I. Find $(a\pm b\pm c)^2$ and $(2a^2-4b^2+3c^2)^2$.

In $(a\pm b\pm c)^2$ there will be nine terms which are either squares or products of two letters : if c is 0 it reduces to $a^2+b^2\pm 2ab$, in which $2ab$ counts as two of the nine terms ; thus we get the square of each letter and twice the product of each pair, positive if the letters have like signs, negative if they have unlike signs.

So
$$(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab,$$
$$(a+b-c)^2=a^2+b^2+c^2-2bc-2ca+2ab,$$
$$(a-b-c)^2=a^2+b^2+c^2+2bc-2ca-2ab,$$

and similarly

$$(2a^2-4b^2+3c^2)^2=4a^4+16b^4+9c^4-24b^2c^2+12c^2a^2-16a^2b^2.$$

Notice that the answers are arranged *according to type*: first the type "square" and then the type "product". Also that the order a, b, c is kept in the products as, term omitting a, term omitting b, term omitting c.

Example II. (i) Given $\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}$ when $2s = a+b+c$, show that
$$16\triangle^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4.$$

Since $s = \frac{1}{2}(a+b+c)$, $s-a = \frac{1}{2}(b+c-a)$, and so for the others.
$$\therefore\ 16\triangle^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

The work is done, mentally, in two stages using in each case the factors of the difference of two squares.
$$(a+b+c)(b+c-a) = (b+c)^2 - a^2 = \quad b^2+c^2-a^2+2bc,$$
$$(c+a-b)(a+b-c) = a^2-(b-c)^2 = -[b^2+c^2-a^2-2bc].$$

The whole product
$$\begin{aligned} 16\triangle^2 &= (2bc)^2 \quad - (b^2+c^2-a^2)^2 \\ &= 4b^2c^2 \quad - \{b^4+c^4+a^4+2b^2c^2-2c^2a^2-2a^2b^2\} \\ &= 2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4. \end{aligned}$$

(ii) Factorise $a^4+b^4+c^4 \pm 2b^2c^2 \pm 2c^2a^2 \pm 2a^2b^2$ for all varieties of the signs.

If all three signs are $+$ or if one of them only is $+$, the expression is a perfect square; $(a^2+b^2+c^2)^2 = a^4+b^4+c^4+2b^2c^2+2c^2a^2+2a^2b^2$ and, for example
$$(a^2+b^2-c^2)^2 = a^4+b^4+c^4-2b^2c^2-2c^2a^2+2a^2b^2.$$

If two of the signs are $+$ and one $-$, we proceed as follows:
$$\begin{aligned} a^4+b^4+c^4+2b^2c^2+2c^2a^2-2a^2b^2 &= (a^2+b^2+c^2)^2 - 4a^2b^2 \\ &= (a^2+b^2+c^2+2ab)(a^2+b^2+c^2-2ab) \\ &= \{(a+b)^2+c^2\}\{(a-b)^2+c^2\}. \end{aligned}$$

Each factor is now the sum of two squares and there are no first degree factors.

If all the signs are $-$ we go backwards through the work of (i) and get
$$\begin{aligned} &a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2 \\ &\qquad = -(a+b+c)(b+c-a)(c+a-b)(a+b-c). \end{aligned}$$

Σ Notation

The expressions in Examples I, II can be written in an abbreviated form by the use of the Σ notation, where Σ is used to mean "the sum of all terms of the type . . . ."

Thus
$$(a+b+c)^2 = \Sigma a^2 + 2\Sigma bc,$$
$$(a^2+b^2+c^2)^2 = \Sigma a^4 + 2\Sigma b^2c^2.$$

The notation Π for products is sometimes used also; thus for the three letters a, b, c we could write
$$\Pi(b+c) \quad \text{for} \quad (b+c)(c+a)(a+b).$$

Example III. Work out (α) $(a+b+c)^3$ and (β) $(a+b+c+d)^3$.

If $c=0$ (α) must reduce to $a^3+3a^2b+3ab^2+b^3$.

Hence (i) $a^3+b^3+c^3$ will be the terms of the type "cube of one letter", for short Σa^3.

(ii) Terms of the type a^2b (square of one letter with first power of another) will each have coefficient 3 and will be

$$3a^2b+3ab^2+3b^2c+3bc^2+3c^2a+3ca^2, \text{ or, for short, } 3\Sigma a^2b$$

[notice that this Σa^2b has six terms, while Σa^3 has only 3].

(iii) Since each of the 3 original factors has 3 terms, there will in the product be 27 terms altogether. So far $3+3\times 6=21$ terms have been accounted for. Each of the remaining terms is abc, so to complete the 27 terms we have $6abc$.

Thus $(a+b+c)^3=\Sigma a^3+3\Sigma a^2b+6abc$.

(β) With 4 letters a, b, c, d there are 12 products of type a^2b, viz. 2 for each pair of letters ab, ac, ad, bc, bd, cd, and 4 products of type abc, viz. those omitting a, b, c, d in turn.

It will be seen that Σa^3 gives 4 terms, $3\Sigma a^2b$ gives 36, and $6\Sigma abc$ gives 24. These terms total 64, which is the correct number 4^3;

$$\therefore\ (a+b+c+d)^3=\Sigma a^3+3\Sigma a^2b+6\Sigma abc.$$

Example IV. Show that $a^3+b^3+c^3-3abc$ is divisible by $a+b+c$ and find the quotient.

Calling the expression $f(a)$, it will be divisible by $a+b+c$ if $f(-b-c)$ is zero.

But $f(-b-c)=-(b+c)^3+b^3+c^3+3(b+c)bc=0$ as required.

The quotient will be homogeneous of the second degree and symmetrical; it will therefore be $p\,\Sigma\, a^2+q\,\Sigma\, bc$ where p and q are constants.

Comparing coefficients of a^3 we see that $p=1$;

$$\therefore\ a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2+qbc+qca+qab).$$

Comparing coefficients of a^2b we see that $0=1+q$ since a^2b comes only from $a\times qab+b\times a^2$. Thus $q=-1$, and finally

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

It may be noted that the expression found as the quotient has no factors in real algebra; if it had factors, it would still have factors when $c=0$, that is a^2+b^2-ab would have factors, which has been shown not to be the case (see p. 37).

In complex algebra, the quotient has factors (involving the complex cube roots of unity), see Examples 103, No. 6.

Note that of the original 3×6 terms in the product, the 12 of the type a^2b cancel each other in pairs, leaving 6 terms, the 3 cubes and $-3abc$.

Example V. Factorise $a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$.

Regarded as a function of a, the expression will have $a-b$ as a factor if $b(b^4-c^4)+b(c^4-b^4)+c(b^4-b^4)=0$, which is the case.

Symmetry shows that $(b-c)(c-a)(a-b)$ is a factor.

The expression is homogeneous and of the fifth degree and symmetrical, so that the remaining factor must be homogeneous, symmetrical and of the second degree.

It is therefore of the form $p(a^2+b^2+c^2)+q(bc+ca+ab)$.

In the identity

$$a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$$
$$\equiv \{(b-c)(c-a)(a-b)\}\{p(a^2+b^2+c^2)+q(bc+ca+ab)\}$$

pick out the term ab^4.

This occurs if $+ab^2$ in the first $\{\quad\}$ is multiplied by $+pb^2$ in the second and nowhere else; whence $p=1$.

Also considering the term in b^3c^2, which does not occur on the left and occurs on the right as $-b^2c\times qbc+bc^2\,.\,pb^2$, we see that $q=p=1$.

Alternatively, p and q may be found more easily by giving special values to a, b and c.

Thus put $a=-1$, $b=0$, $c=1$.

We get $1+0+1=(-1)2(-1)(2p-q)$ or $4p-2q=2$.

Put $a=0$, $b=1$, $c=2$,

$$0+16-2=(-1)2(-1)(5p+2q) \quad \text{or} \quad 10p+4q=14.$$

These equations give $p=q=1$ as before.

$$\therefore\ a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$$
$$=(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab).$$

Note that the first part of the argument will apply to

$$a(b^n-c^n)+b(c^n-a^n)+c(a^n-b^n)$$

for all integral values of n.

These expressions are not symmetrical, but possess what is called *cyclic symmetry*; that is to say they are unaltered by the change a to b, b to c, c to a; though they would be altered, in sign, by the change a to b, b to a alone.

However, the factors $(b-c)(c-a)(a-b)$ introduce this cyclic property, and the remaining factor will be symmetrical in the strict sense.

Examples 50

1. Write down the squares of $b^2+c^2-a^2$ and of $a^2-b^2-c^2$.

2. Factorise (i) $a^2+b^2-c^2+2ab$ (ii) $a^2+b^2-c^2-2ab$; (iii) $9a^2-25b^2-4c^2+20bc$.

3. If $s=\frac{1}{2}(a+b+c)$ work out $4[(s-a)^2+(s-b)^2+(s-c)^2]$ in terms of a, b, c.

4. Work out or write down $(a-b-c)^2$ and $(a-b-c)^3$.

5. Give the worked-out form of $\Sigma a \times (\Sigma a^2 + \Sigma bc)$ for the three letters a, b, c.

6. Simplify the expression found for Δ^2 in Example II if it is given that $a^2 = b^2 + c^2$.

7. Show that $(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2) = 0$, which is the condition that the triangle with sides a, b, c, should be right-angled, works out to

$$-\Sigma a^6 + \Sigma a^4 b^2 - 2a^2 b^2 c^2,$$

in which note that the first Σ implies 3 terms but the second Σ 6 terms.

[It is only necessary to count up in the product the way in which a^6, a^4b^2 and $a^2b^2c^2$ can occur ; symmetry does the rest.]

8. Use the factors of $a^3 + b^3 + c^3 - 3abc$ to obtain the factors of

(i) $a^3 - b^3 + c^3 + 3abc$; (ii) $a^3 - b^3 - c^3 - 3abc$;
(iii) $a^3 + 8b^3 + 27c^3 - 18abc$; (iv) $64a^3 - 8b^3 + 27c^3 + 72abc$.

9. What is the distance between the points (p, q, r) and $(-2p, q, 3r)$ in solid geometry? If the line joining them subtends a right angle at the origin, prove $q^2 = 2p^2 - 3r^2$.

10. A is the point $(p, -q, r)$; B the point $(p, q, -r)$. If AB subtends an angle of 120° at the origin, prove that $3p^2 = q^2 + r^2$.

11. Show that the condition that $x^2 - (a^2 + b^2 + c^2)x + b^2c^2 + c^2a^2 + a^2b^2$ should be a perfect square can be put in the form

$$\{a^2 - (b + c)^2\}\{a^2 - (b - c)^2\} = 0,$$

and that this is impossible if a, b, c are the sides of a triangle.

12. Show that $(p + r)^2 + (q + r)^2 - (p + q + r)^2 = r^2 - 2pq$.

Hence show that if $2pq = r^2$ and all the letters stand for positive integers, we obtain sides of a Pythagorean triangle (right-angled triangle with integral sides). Consider the special case when $p = 9$, $q = 2$.

13. Find the factors of

(i) $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$;
(ii) $a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2)$.

14. Prove the following identities (the letters a, b, c being involved) :

(i) $\Sigma bc(b - c) = (a - c)(b - a)(c - b) = \Sigma a^2(b - c)$;
(ii) $\Sigma\{a^2(b + c - a)\} = (a + b - c)(b + c - a)(c + a - b) + 2abc$;
(iii) $\Sigma\{a(b - c)^3\} = (b - c)(c - a)(a - b)(a + b + c)$;
(iv) $\Sigma\{a(b^2 + c^2)\} = (b + c)(c + a)(a + b) - 2abc$.

15. Show that : (i) $\Sigma a \times \Sigma b^2c^2 = \Sigma a^3b^2 + abc\Sigma ab$;
(ii) $\Sigma bc(b^2 - c^2) = -(a - b)(b - c)(c - a)(a + b + c)$.

Miscellaneous Examples 51

1. Factorise:

(i) $x^6 - y^3z^3$; (ii) $x^4 - 13x^2 - 14$;
(iii) $x^4 - 0{\cdot}0016$; (iv) $x^4 - 81$;
(v) $x^4 + x^2y^2 + y^4$; (vi) $x^4 + 1/x^4 - 2$;
(vii) $x^3 - ax^2 - a^2x + a^3$; (viii) $x^4 - 12x^2 + 16$.

2. Factorise:

(i) $12x^2 + xy - 6y^2 + 14x + 19y - 10$;
(ii) $6x^2 - 5xy - 4y^2 + 5x + 8y - 4$;
(iii) $x^2 - 4xy + 3y^2 - 2y - 1$;
(iv) $x^2 - y^2 + 6x + 9$;
(v) $6x^2 - 5xy - 6y^2 - x + 8y - 2$.

3. Factorise:

(i) $a^3(b-c) + b^3(c-a) + c^3(a-b)$;
(ii) $a^4(b-c) + b^4(c-a) + c^4(a-b)$;
(iii) $bc(b^2-c^2) + ca(c^2-a^2) + ab(a^2-b^2)$;
(iv) $(x+y+z)^3 - x^3 - y^3 - z^3$;
(v) $(y-z)^5 + (z-x)^5 + (x-y)^5$.

4. Find the remainder when:

(i) $x^3 - 2x^2 + 11x + 17$ is divided by $x - 2$;
(ii) $3x^3 + x^2 + 5$ is divided by $x + 2$;
(iii) $x^4 - 3x^2 + 7$ is divided by $2x - 3$;
(iv) $x^2 + 3x + 1$ is divided by $3x + 1$.

5. Solve the equations:

(i) $x^3 - 7x - 6 = 0$ given that one root is -1;
(ii) $2x^3 + x^2 - 8x - 4 = 0$ given that one root is 2;
(iii) $x^3 + 4x^2 - 11x + 6 = 0$;
(iv) $x^3 - 4x + 3 = 0$.

6. Factorise:

(i) $4x^4 - 17x^2 + 4$;
(ii) $3x^4 - 5x^3 + 3x - 1$;
(iii) $6x^2 - 8xy - 8y^2 + 5x + 14y - 6$;
(iv) $(a+b+c)^3 - a^3 - b^3 - c^3 + 3abc$.

7. Find the values of a, b, c which make

$$x^4 - ax^3 + bx^2 - cx + 6$$

divisible by $x - 1$, $x - 2$ and $x - 3$. (O. & C.)

8. When a polynomial $P(x)$ is divided by $x^2 - 3x + 2$ prove that the remainder is $x\{P(2) - P(1)\} + \{2P(1) - P(2)\}$ and that the term independent of x in the quotient is

$$\tfrac{1}{2}\{P(0) - 2P(1) + P(2)\}.$$ (N.)

9. Prove that if a polynomial in x, $\phi(x)$ is divided by x^2-a^2, the remainder is of the form $Lx+M$ where

$$L=\frac{1}{2a}\{\phi(a)-\phi(-a)\}, \quad M=\tfrac{1}{2}\{\phi(a)+\phi(-a)\}.$$

Show that for the expression

$$x^5+3x^4a+2x^3a^2-x^2a^3-3xa^4-2a^5$$

$L=0$, $M=0$. Hence, or otherwise, factorise the expression. (L.)

10. Prove that, if $f(x)$ is a polynomial in x, the remainder on dividing it by $x-a$ is $f(a)$.

If the quotient on dividing by $x-a$ is $g(x)$ and on dividing by $x-b$ is $h(x)$, prove that $g(b)=h(a)=\dfrac{f(a)-f(b)}{a-b}$.

11. If $f(x)\equiv a_0x^n+a_1x^{n-1}+\dots+a_n$ prove that when $f(x)$ is divided by $(x-p)(x-q)$ the remainder is

$$[(x-q)f(p)-(x-p)f(q)]/(p-q).$$

Find by inspection a root of the equation $x^3-2x+1=0$ and hence solve the equation completely. (N.)

12. If k is an odd positive integer, prove that

(i) $(a+b+c)^k-a^k-b^k-c^k$ is divisible by $(b+c)(c+a)(a+b)$;

(ii) $(b-c)^k+(c-a)^k+(a-b)^k$ is divisible by $(b-c)(c-a)(a-b)$.

Factorise $(a+b+c)^5-a^5-b^5-c^5$.

13. Factorise completely the expression

(i) $a^3+b^3+c^3+a(a-b)(c-a)+b(b-c)(a-b)+c(c-a)(b-c)+5abc$,

and, by putting $(x-3)(x-6)=y$, or otherwise, the expression

(ii) $4(x-2)(x-3)(x-6)(x-9)-5x^2$. (L.)

14. Find the possible values of p in order that

$$4x^2+4x-3 \text{ and } 4x^3-8x^2+px-4$$

may have a common factor. (L.)

15. If x^3 is expressed in the form

$$a+b(x-1)+c(x-1)(x-2)+d(x-1)(x-2)(x-3)$$

find the values of a, b, c, d. (N.)

16. By means of the remainder theorem, or otherwise, find the roots of the equation $x^3+5x^2-2x-24=0$ being given that all the roots are integral.

If $p_0x^3+p_1x^2+p_2x+p_3\equiv(x-a)(q_0x^2+q_1x+q_2)+R$, prove that $q_0=p_0$, $q_1=aq_0+p_1$, $q_2=aq_1+p_2$, $R=aq_2+p_2$ and express q_1, q_2, R in terms of a, p_0, p_1, p_2, p_3. (L.)

17. Find the values of a and b, different from c and d, which make the equation

$$3(x+a)^2+2(x+b)^2=3(x+c)^2+2(x+d)^2$$

true for all values of x. (L.)

18. Prove that $(x-1)^2$ is a common factor of

$$px^{p+1}-(p+1)x^p+1 \text{ and } x^p-px+p-1$$

where p is any positive integer. (L.)

19. Prove that, if $2s=a+b+c$,

(i) $(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2$;

(ii) $(s-a)^3+(s-b)^3+(s-c)^3-s^3=-3abc$. (L.)

20. If $a+b+c=0$, prove that

(i) $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)=abc(b-c)(c-a)(a-b)$;

(ii) $a^4+b^4+c^4=2(bc+ca+ab)^2$. (L.)

21. If $f(x)=ax^2+bx+c$, prove that

$$f(x+mb)-2f(x)+f(x-mb),$$

where m is any constant, is independent of x. (L.)

22. If $x-\dfrac{1}{x}=y$, prove that $x^3-\dfrac{1}{x^3}=y^3+3y$ and express $x^5-\dfrac{1}{x^5}$ in terms of y. (L.)

23. Find values of a and b which make the expression

$$x^4+6x^3+13x^2+ax+b$$

a perfect square. (N.)

24. Give the remainder (free from x) when x^7+a^7 is divided by $x-a$. If a constant b is chosen so that $x^7+y^7-bx^3y^4$ is divisible by $x-y$ find the value of b and the quotient. (O. & C.)

CHAPTER VI

PARTIAL FRACTIONS, CHANGE OF VARIABLE

Partial Fractions: Addition

THE identity $$\frac{x^2-2x+2}{(x-2)(x-3)^2} \equiv \frac{2}{x-2} - \frac{1}{x-3} + \frac{5}{(x-3)^2}$$

in which a single fraction is said to be equal to the sum of what are called its "partial fractions" can be proved either by adding together the partial fractions, or by splitting up the single fraction to obtain the partial fractions.

For several purposes the expression in partial fractions is more useful than the other, especially for integrating or differentiating.

Integrating gives $2\log(x-2) - \log(x-3) - 5/(x-3)$, while differentiating gives

$$-2/(x-2)^2 + 1/(x-3)^2 - 10/(x-3)^3,$$

and each of these processes would be less easy to apply to the single fraction.

The methods used to express a single fraction in partial fractions form the chief part of this chapter, but as the student has probably tackled only easy examples of the adding-together process, some examples of this will be given first.

Example I. Express as a single fraction

$$E \equiv \frac{2x-1}{x^2-4} + \frac{x+3}{x^2-x-2} - \frac{x-3}{x^2+3x+2}.$$

[The first step is to express the denominators in factors.]

$$E = \frac{2x-1}{(x+2)(x-2)} + \frac{x+3}{(x-2)(x+1)} - \frac{x-3}{(x+2)(x+1)}.$$

[The L.C.M. of the denominators is seen to be $(x+2)(x+1)(x-2)$]

$$\therefore\ E = \frac{(2x-1)(x+1)+(x+3)(x+2)-(x-3)(x-2)}{(x+2)(x+1)(x-2)}$$

$$= \frac{2x^2+x-1+x^2+5x+6-x^2+5x-6}{(x+2)(x+1)(x-2)}$$

$$= \frac{2x^2+11x-1}{(x+2)(x+1)(x-2)}.$$

Example II. Express as a single fraction

$$E=\frac{3x^2+2x+5}{x^2+1}-\frac{3x^2-4x+7}{x^2-1}.$$

[x^2+1 has no factors ; it is no help to factorise x^2-1.]

$$E=\frac{(3x^2+2x+5)(x^2-1)-(3x^2-4x+7)(x^2+1)}{x^4-1}$$

$$=\frac{3x^4+2x^3+2x^2-2x-5-(3x^4-4x^3+10x^2-4x+7)}{x^4-1}$$

$$=\frac{6x^3-8x^2+2x-12}{x^4-1}.$$

This example differs from the first in that the given fractions are not algebraically *proper* fractions, a "proper fraction" in algebra meaning one in which the numerator is *lower in degree* than the denominator. In such a case it often saves trouble to separate the integral and fractional parts at once.

In this case $E=\frac{3(x^2+1)+2x+2}{x^2+1}-\frac{3(x^2-1)-4x+10}{x^2-1}$

$$=3+\frac{2x+2}{x^2+1}-3+\frac{4x-10}{x^2-1}$$

$$=\frac{(2x+2)(x^2-1)+(4x-10)(x^2+1)}{x^4-1},$$

which will give the answer as before.

Example III. Simplify

$$\frac{a(a^2+x^2)}{(a-b)(a-c)}+\frac{b(b^2+x^2)}{(b-c)(b-a)}+\frac{c(c^2+x^2)}{(c-a)(c-b)}.$$

The L.C.M. is $(b-c)(c-a)(a-b)$, and each denominator has one sign different from that in the L.C.M.

The sum $=\frac{1}{(b-c)(c-a)(a-b)}\,\Sigma\{-a(a^2+x^2)(b-c)\}$.

In this Σ, the coefficient of x^2 is $-\Sigma\, a(b-c)$ which is 0, and the rest is $-\Sigma\, a^3(b-c)$, which can be shown (see Examples 50, No. 3 (i)) to be

$$(b-c)(c-a)(a-b)(a+b+c).$$

$\therefore$ the sum is $a+b+c$.

Examples 52

1. Express as a single fraction :

(i) $\frac{x+3}{x^2-4}-\frac{4-x}{x^2-5x+6}+\frac{2x}{6+x-x^2}$;

(ii) $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$.

2. Simplify $\dfrac{3x}{x^2-4}+\dfrac{2x+5}{x^2+x-2}-\dfrac{4x-7}{x^2-3x+2}$.

3. Simplify $\dfrac{3x+4}{2x^2+x-1}+\dfrac{4x+3}{2x^2-5x+2}+\dfrac{1}{x+1}$.

4. Simplify $\dfrac{1}{x-1}-\dfrac{x}{x^2-1}-\dfrac{x^2}{x^4-1}-\dfrac{x^4}{x^8-1}$.

5. Simplify $\dfrac{1}{x^2-ax+a^2}-\dfrac{1}{x^2+ax+a^2}-\dfrac{a}{x^3-a^3}-\dfrac{a}{x^3+a^3}$.

6. Simplify $\dfrac{1}{x-a}-\dfrac{x^2+a^2}{x^3-a^3}+\dfrac{1}{x+a}-\dfrac{x^2+a^2}{x^3+a^3}$.

7. Simplify $\dfrac{x+y}{(x-y)(z-y)}+\dfrac{2z}{(x-y)(x-z)}-\dfrac{y+z}{(y-z)(y-x)}$.

8. Simplify $\dfrac{a(a^2+d^2)}{(a-b)(a-c)}+\dfrac{b(b^2+d^2)}{(b-c)(b-a)}+\dfrac{c(c^2+d^2)}{(c-a)(c-b)}$.

9. Show that $\displaystyle\sum_{a,b,c,d}\frac{a(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}=x$.

10. Show that $\displaystyle\sum_{x,y,z}\frac{(x+p)(x+q)}{(x-y)(x-z)(x+h)}=\frac{(h-p)(h-q)}{(x+h)(y+h)(z+h)}$.

11. Show that $\displaystyle\sum_{a,b,c}\frac{1}{1+a^{a-b}+x^{a-c}}=1$.

12. Add $\dfrac{a^2+b^2-ab+a+b+1}{a-b-1}$ to $\dfrac{a^2+b^2+ab+a-b+1}{a+b-1}$.

13. If $x=a+b+\dfrac{(a-b)^2}{4(a+b)}$ and $y=\dfrac{a+b}{4}+\dfrac{ab}{a+b}$, prove that

$$(x-a)^2-(y-b)^2=b^2.$$

14. Show that the sum of two *algebraically proper* fractions is a proper fraction. [This is, of course, not true in arithmetic.]

Finding Partial Fractions

In discussing Partial Fractions we begin with examples.

Example I. Assuming that $\dfrac{2x-1}{(x-2)(x-1)}\equiv\dfrac{a}{x-2}+\dfrac{b}{x-1}$, find the constants a and b.

The fractions on the right are the "*partial fractions*", which together are identically equal to the fraction on the left.

Multiplying both sides by $(x-2)(x-1)$ we get

$$2x-1\equiv a(x-1)+b(x-2). \qquad \text{(I)}$$

As this is to be an identity we can equate the coefficients of x and the constant or "*absolute*" term and get

$$\left.\begin{aligned} a+b&=2 \\ -a-2b&=-1 \end{aligned}\right\} \begin{aligned}&\text{which give} \\ &b=-1,\ a=3.\end{aligned}$$

Hence
$$\frac{2x-1}{(x-2)(x-1)} \equiv \frac{3}{x-2} - \frac{1}{x-1}.$$

This can be verified by adding the fractions on the right.

As equation (I) is an identity, we can proceed in another way, which is usually easier; the necessary equations are obtained by giving special values to x, *so as to get the coefficients one at a time.*

Thus put $\left.\begin{aligned} x=1 \text{ in (I); then } 1&=b(-1) \\ x=2 \text{ in (I); then } 3&=a\,.\,1 \end{aligned}\right\}$ giving the result as before.

Caution

It is very important to realise that the validity of this second method depends entirely on the correctness of the original assumption, and if that is not correct a wrong result is obtained.

If, for example, we suppose that $\dfrac{x^2}{(x-2)(x-1)} \equiv \dfrac{a}{x-2} + \dfrac{b}{x-1}$, multiplying up gives $x^2 \equiv a(x-1)+b(x-2)$. ....(II)

In this put $x=1$; then $1=0+b(-1)$:

put $x=2$; then $4=a\,.\,1+0$,

from which we should conclude that

$$\frac{x^2}{(x-2)(x-1)} \equiv \frac{4}{(x-2)} - \frac{1}{x-1}.$$

This conclusion is *false*, although there has been no hitch in the work.

Our original assumption was impossible (as we should have seen at once if we had tried to equate coefficients in (II)).

The correct assumption would have been

$$\frac{x^2}{(x-2)(x-1)} \equiv c + \frac{a}{x-2} + \frac{b}{x-1},$$

(II) being replaced by $x^2 = c(x^2-3x+2)+a(x-1)+b(x-2)$.

Here from coefficients of x^2 we see that $c=1$, and then putting

$$x=1 \text{ gives } 1=c\,.\,0+a\,.\,0+b(-1),$$
$$x=2 \text{ gives } 4=c\,.\,0+a\,.\,1+b\,.\,0,$$

so that the correct result is

$$\frac{x^2}{(x-2)(x-1)} \equiv 1 + \frac{4}{3(x-2)} - \frac{1}{x-1}.$$

This example illustrates the fact that when the degree of the numerator is *not less* than the degree of the denominator, i.e. when the original

fraction is not an (algebraically) *proper* fraction, there will be an integral function as well as the partial fractions, *if* they are to be *proper* fractions.

Thus $\dfrac{x^3}{(x-2)(x-1)}$ could be expressed as $ax+b+\dfrac{c}{x-2}+\dfrac{d}{x-1}$.

One way of finding the " $ax+b$ " part is to divide x^3 by x^2-3x+2 as on the right and obtain

$$\begin{array}{r|rrrr|l} 1-3+2 & 1 & & & & 1+3 \\ \cline{1-1} & 1 & -3 & +2 & & \\ \cline{2-4} & & 3 & -2 & & \\ & & 3 & -9 & +6 & \\ \cline{3-5} & & & 7 & -6 & \end{array}$$

$$x+3+\frac{7x-6}{(x-2)(x-1)}$$

Using the method of Example I we get

$$7x-6=c(x-1)+d(x-2).$$

Putting $x=2$ gives $c=8$ and putting $x=1$ gives $d=-1$;

$$\therefore \quad \frac{x^3}{(x-2)(x-1)}=x+3+\frac{8}{x-2}-\frac{1}{x-1}.$$

Partial Fractions for a Proper Fraction

In the next examples we shall suppose that the original fraction is a *proper* fraction, i.e. with the degree of the numerator less than that of the denominator.

In this case the partial fraction corresponding to a linear factor in the denominator can be found by a simple rule (sometimes called the " cover-up " rule) which saves trouble if used at once.

The coefficient of $\dfrac{1}{ax-b}$ is the result of putting $x=\dfrac{b}{a}$ in the rest of the expression (i.e. with $ax-b$ " covered up ").

Example II. Put $\dfrac{x+1}{(x-5)(2x-3)}$ into partial fractions.

The coefficient of $\dfrac{1}{x-5}$ will be $\dfrac{5+1}{2\cdot 5-3}=\dfrac{6}{7}$.

The coefficient of $\dfrac{1}{2x-3}$ will be $\dfrac{\frac{3}{2}+1}{\frac{3}{2}-5}=-\dfrac{5}{7}$.

$\therefore$ the fraction $=\dfrac{6}{7(x-5)}-\dfrac{5}{7(2x-3)}$.

To prove this, suppose the fraction $=\dfrac{A}{x-5}+\dfrac{B}{2x-3}$.

Then multiplying by $x-5$ gives $\dfrac{x+1}{2x-3}=A+\dfrac{B(x-5)}{2x-5}$. ....(i)

while multiplying by $2x-3$ gives $\dfrac{x+1}{x-5}=\dfrac{A(2x-3)}{x-5}+B$. ....(ii)

Putting $x=5$ in (i) gives $\dfrac{6}{10-3}=A$,

$x=\frac{3}{2}$ in (ii) gives $\dfrac{\frac{3}{2}+1}{\frac{3}{2}-5}=B$.

These are the results given by the rule.

Example III. Express $\dfrac{7x-1}{x^3-1}$ in partial fractions.

Assuming that

$$\frac{7x-1}{(x-1)(x^2+x+1)}\equiv\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1},$$

since (x^2+x+1) has no factors, we get

$$\frac{7x-1}{x^2+x+1}\equiv A+\frac{(x-1)(Bx+C)}{x^2+x+1}.$$

Putting $x=1$ gives $\dfrac{7-1}{1+1+1}=A$, i.e. $A=2$.

Now $\dfrac{7x-1}{x^3-1}-\dfrac{2}{x-1}=\dfrac{(7x-1)-2(x^2+x+1)}{(x-1)(x^2+x+1)}=\dfrac{-2x^2+5x-3}{(x-1)(x^2+x+1)}$.

The numerator of the last fraction $=(x-1)(-2x+3)$ and *the fact that* $(x-1)$ *is a factor is a check on the previous work.*

$$\therefore\ \frac{7x-1}{x^3-1}\equiv\frac{2}{x-1}+\frac{-2x+3}{x^2+x+1}.$$

Alternatively, having found A, the B and C of the original assumption may be found from $7x-1\equiv 2(x^2+x+1)+(Bx+C)(x-1)$ by equating coefficients.

Example IV. Find the partial fraction corresponding to the factor $(2x+1)$ when $\dfrac{10x^2-23x+11}{(2x+1)(x-2)^2}$ is put into partial fractions.

To find this, cover up the $(2x+1)$ and write $x=-\frac{1}{2}$ in $\dfrac{10x^2-23x+11}{(x-2)^2}$.

The result is $\dfrac{10/4+23/2+11}{(-5/2)^2}=\dfrac{10+46+44}{25}=4$.

So the required fraction is $\dfrac{4}{2x+1}$, which could be justified as in Example II.

If the partial fractions corresponding to $(x-2)^2$ are required, there are two ways of proceeding:

(i) Subtract $\dfrac{4}{2x+1}$ from the original fraction to give a fraction with $(2x+1)(x-2)^2$ as denominator.

$$\text{The numerator} = 10x^2 - 23x + 11 - 4(x-2)^2$$
$$= 6x^2 - 7x - 5$$
$$= (2x+1)(3x-5).$$

That $(2x+1)$ is a factor is a check on the previous work.

$\therefore$ the original fraction $= \dfrac{4}{2x+1} + \dfrac{3x-5}{(x-2)^2}$,

and since $3x - 5 = 3(x-2) + 1$,

the original fraction $= \dfrac{4}{2x+1} + \dfrac{3}{x-2} + \dfrac{1}{(x-2)^2}$.

This is a more convenient form if we have either to differentiate or to integrate.

(ii) We can begin by assuming that the original fraction is equivalent either to $\dfrac{A}{2x+1} + \dfrac{Bx+C}{(x-2)^2}$ or to $\dfrac{A}{2x+1} + \dfrac{P}{x-2} + \dfrac{Q}{(x-2)^2}$.

Multiplying up and equating coefficients will determine the unknown quantities. [This is left as an exercise for the student.]

Example V. *The General Case*

If $\dfrac{f(x)}{(ax-b)F(x)}$ is a proper fraction and $(ax-b)$ is not a factor of $F(x)$ so that $F\left(\dfrac{b}{a}\right) \neq 0$, find the partial fraction corresponding to $(ax-b)$.

The partial fractions will be $\dfrac{A}{ax-b} + \dfrac{P(x)}{F(x)}$ where $P(x)$ is some function of x.

Hence multiplying by $(ax-b)$,

$$\frac{f(x)}{F(x)} = A + \frac{(ax-b)P(x)}{F(x)}.$$

In this put $x = \dfrac{b}{a}$; the second term on the right is zero, and we get

$$\frac{f\left(\dfrac{b}{a}\right)}{F\left(\dfrac{b}{a}\right)} = A.$$

This is the "cover-up" rule.

Note. If the denominator is $Q(x)$ where

$$Q(x) = (x-a_1)(x-a_2)\ldots(x-a_n),$$

then its derivative $Q'(x)$ is $\Sigma(x-a_2)(x-a_3)\ldots(x-a_n)$, this summation consisting of the n products got by leaving out one factor at a time.

Hence, for $\dfrac{f(x)}{Q(x)}$, the partial fraction corresponding to $(x-a_1)$ is

$$\frac{f(a_1)}{Q'(a_1)}\cdot\frac{1}{x-a_1},$$

and similarly for the others.

Examples 53. Express the following fractions in partial fractions :

1. $\dfrac{2x+8}{(x+5)(x+3)}$.
2. $\dfrac{2}{(x+7)(x+9)}$.
3. $\dfrac{5-x}{(3x+1)(x+3)}$.
4. $\dfrac{x-7}{x^2+6x+5}$.
5. $\dfrac{4x^2-26x+46}{(x-2)(x-3)(x-4)}$.
6. $\dfrac{2x}{x^2-a^2}$.
7. $\dfrac{x}{(x-2)(x-3)}$.
8. $\dfrac{x^2}{(x-2)(x-3)}$.
9. $\dfrac{2x^2}{x^2-a^2}$.
10. $\dfrac{8-x-x^2}{x^3+4x}$.
11. $\dfrac{3-x+2x^2}{(x+1)(x^2+1)}$.
12. $\dfrac{x}{(x+1)(x^2+1)}$.
13. $\dfrac{x}{(x-1)(x^2-1)}$.
14. $\dfrac{5x^2-2x+3}{(x-1)(x^2+1)}$.
15. $\dfrac{x^2+4}{x(x^2+x+1)}$.
16. $\dfrac{2x-1}{x^3+1}$.
17. $\dfrac{x^2+4x+7}{(x+1)^2(x+3)}$.
18. $\dfrac{1}{(x+2)^2(2x+3)}$.
19. $\dfrac{7-4x+5x^2}{(1-x)^2(1+x^2)}$.
20. $\dfrac{x}{(x-7)(x^2+1)}$.
21. $\dfrac{x^2}{(1+x^2)(2+x^2)}$.

Repeated Factors in Denominator

Example I. Express $\dfrac{x^2}{(x-2)(x-1)^3}$ in the form $\dfrac{a}{x-2}+\dfrac{bx^2+cx+d}{(x-1)^3}$ and in the form $\dfrac{a}{x-2}+\dfrac{p}{x-1}+\dfrac{q}{(x-1)^2}+\dfrac{r}{(x-1)^3}$.

Using the previous formula, or (if preferred) clearing fractions and putting $x=2$ we get $a=2^2/(2-1)^3=4$;

$\therefore\ \dfrac{x^2}{(x-2)(x-1)^3}-\dfrac{4}{x-2}$ will be a fraction with denominator $(x-1)^3$.

Now
$$\begin{aligned}x^2-4(x-1)^3 &= -4x^3+13x^2-12x+4\\ &= -4x^3+8x^2+5x^2-10x-2x+4\\ &=(x-2)(-4x^2+5x-2);\end{aligned}$$

$$\therefore\ \frac{x^2}{(x-2)(x-1)^3}=\frac{4}{x-2}-\frac{4x^2-5x+2}{(x-1)^3}.$$

To get the second required form, we must find p, q, r so that

$$p(x-1)^2+q(x-1)+r=-4x^2+5x-2.$$

This can be done by equating coefficients (but see Example II):

$$\begin{array}{lr} (x^2) & p = -4 \\ (x) & -2p + q = 5 \\ (\text{constant}) & p - q + r = -2 \end{array} \left.\vphantom{\begin{array}{l}a\\b\\c\end{array}}\right\} \begin{array}{l} p = -4, \\ q = -3, \\ r = -1. \end{array}$$

$$\therefore \text{ the fraction} = \frac{4}{x-2} - \frac{4}{x-1} - \frac{3}{(x-1)^2} - \frac{1}{(x-1)^3}.$$

Example II. If $\dfrac{2x^3-4x^2-15x+18}{(x-3)^4} = \dfrac{a}{x-3} + \dfrac{b}{(x-3)^2} + \dfrac{c}{(x-3)^3} + \dfrac{d}{(x-3)^4}$ find a, b, c, d.

Clearing fractions we get

$$2x^3 - 4x^2 - 15x + 18 \equiv a(x-3)^3 + b(x-3)^2 + c(x-3) + d. \quad \text{.......(A)}$$

The question can be finished as in Example I, but the following method will be found quicker, once it is known.

Looking at the right-hand side of (A) we see that if this expression is divided by $(x-3)$ the remainder is d and the quotient is

$$a(x-3)^2 + b(x-3) + c.$$

A second division by $(x-3)$ gives c as remainder and the quotient is $a(x-3)+b$, so that a third division gives b as remainder and a as quotient.

We therefore *divide the left-hand side repeatedly by* $(x-3)$.

Detaching coefficients and not writing down the divisor:

$$\begin{array}{r|r|r|l} 2-4-15+18 & 2+2-\ \ 9 & 2+\ 8 & 2 \\ \underline{2-6}\qquad\quad & \underline{2-6}\quad\ \ & \underline{2-\ 6} & \\ 2-15\quad\ \ & 8-\ \ 9 & 14\ \ & \\ \underline{2-\ \ 6}\quad\ \ & \underline{8-24} & & \\ -\ \ 9+18 & 15 & & \\ \underline{-\ \ 9+27} & & & \\ -\ \ 9 & & & \end{array} \qquad \text{Thus} \quad \begin{array}{l} a = 2 \\ b = 14 \\ c = 15 \\ d = -9 \end{array}$$

Change of Variable

It should be understood that what was done in Example II was to *change the variable* in $2x^3-4x^2-15x+18$ from x to x' where $x' = x-3$, with the result

$$2x^3 - 4x^2 - 15x + 18 = 2x'^3 + 14x'^2 + 15x' - 9.$$

From the point of view of the graph of the function, it is a case of *moving the origin* along the x-axis to the place where $x=3$. If this is done and x' is the new abscissa, we have $x' = x-3$ or, if preferred, $x = x'+3$.

Example III. Express $5x^4 - 2x^3 + x - 7$ as a function of x' where $x' = x+2$.

Detach coefficients and divide by $x+2$.

$$\begin{array}{l|l|l|l|l}
5-\ 2+\ 0+\ 1-\ 7 & 5-12+24-\ 47 & 5-22+68 & 5-32 & 5\\
\underline{5+10} & \underline{5+10} & \underline{5+10} & \underline{5+10} & \\
\quad -12+\ 0 & \quad -22+24 & \quad -32+68 & \quad -42 & \\
\quad \underline{-12-24} & \quad \underline{-22-44} & \quad \underline{-32-64} & & \\
\qquad\quad 24+\ 1 & \qquad\quad 68-\ 47 & \qquad\quad 132 & & \\
\qquad\quad \underline{24+48} & \qquad\quad \underline{68+136} & & & \\
\qquad\qquad -47-\ 7 & \qquad\qquad -183 & & & \\
\qquad\qquad \underline{-47-94} & & & & \\
\qquad\qquad\qquad 87 & & & &
\end{array}$$

Result : $5x^4-2x^3+x-7=5x'^4-42x'^3+132x'^2-183x'+87$.

This result can, of course, be obtained by equating coefficients as in Example I. The student may like to try this.

Note. Finding partial fractions for a fraction whose denominator contains more than one linear factor which is repeated can be troublesome. If a repeated quadratic factor occurs, the work is very troublesome in real algebra, but in complex algebra (see Chap. XI) the quadratic factor always has linear factors.

Example IV. Find a polynomial $P(x)$ of degree 2 (or less) which has values 2, 6, 54 when x is 1, 2, 5 respectively.

The partial fractions for $\dfrac{P(x)}{(x-1)(x-2)(x-5)}$

are
$$\frac{P(1)}{(1-2)(1-5)}\cdot\frac{1}{x-1}+\frac{P(2)}{(2-1)(2-5)}\cdot\frac{1}{x-2}+\frac{P(5)}{(5-1)(5-2)}\cdot\frac{1}{x-5}$$
$$=\frac{2}{4}\cdot\frac{1}{x-1}+\frac{6}{-3}\cdot\frac{1}{x-2}+\frac{54}{12}\cdot\frac{1}{x-5};$$
$$\therefore\ P(x)=\tfrac{1}{2}(x-2)(x-5)-2(x-1)(x-5)+\tfrac{9}{2}(x-1)(x-2).$$

This result is easily seen to be correct.

It could also be obtained by assuming that
$$P(x)=A(x-2)(x-5)+B(x-1)(x-5)+C(x-1)(x-2)$$
and substituting $x=1,\ 2,\ 5$ in turn.

Note. This is generalised as follows.

If $P(x)$ is to be a polynomial, of degree at most $n-1$, having the values $A_1, A_2, \ldots A_n$ corresponding to $x=a_1, a_2, \ldots a_n$ respectively, take
$$Q(x)=(x-a_1)(x-a_2)\ldots(x-a_n).$$

Then
$$\frac{P(x)}{Q(x)}=\sum_{r=1}^{n}\frac{P(a_r)}{Q'(a_r)}\cdot\frac{1}{x-a_r}\ \text{and}\ P(a_r)\equiv A_r,$$
so that
$$P(x)=\sum_{r=1}^{n}\frac{A_r}{Q'(a_r)}\cdot\frac{Q(x)}{x-a_r}.$$

This result is *Lagrange's Interpolation Formula.*

Harder Examples

Examples V and VI are harder.

Some may prefer to omit them, and Nos. 18, 19, 20 of Examples 54.

Example V. Find partial fractions for $\dfrac{2x^2}{(x-3)^3(x^2-x-4)}$.

Put $x-3=y$, then $x=y+3$, $2x^2=2(y^2+6y+9)$,

$$x^2-x-4=y^2+6y+9-y-3-4=y^2+5y+2.$$

The expression $=\dfrac{2(y^2+6y+9)}{y^3(y^2+5y+2)}$.

Divide, in *ascending* powers of y, $9+6y+y^2$ by $2+5y+y^2$.

The work at side shows that:

$$(9+6y+y^2)\div(2+5y+y^2)$$

$$=\frac{9}{2}-\frac{33}{4}y+\frac{151}{8}y^2-\frac{689y^3+151y^4}{8(2+5y+y^2)}.$$

$$\begin{array}{r|llll|l}
2+5+1 & 9+ & 6+1 & & & \frac{9}{2}-\frac{33}{4}+\frac{151}{8} \\
 & 9+ & \frac{45}{2}+\frac{9}{2} & & & \\ \hline
 & & -\frac{33}{2}-\frac{7}{2} & & & \\
 & & -\frac{33}{2}-\frac{165}{4}-\frac{33}{4} & & & \\ \hline
 & & & \frac{151}{4}+\frac{33}{4} & & \\
 & & & \frac{151}{4}+\frac{755}{8}+\frac{151}{8} & & \\ \hline
 & & & & -\frac{689}{8}-\frac{151}{8} &
\end{array}$$

Hence the original fraction

$$=2\left\{\frac{9}{2y^3}-\frac{33}{4y^2}+\frac{151}{8y}-\frac{689+151y}{8(2+5y+y^2)}\right\}$$

$$=\frac{9}{(x-3)^3}-\frac{33}{2(x-3)^2}+\frac{151}{4(x-3)}-\frac{151x+236}{4(x^2-x-4)}.$$

Example VI. Consider an algebraically proper fraction such as

$$\frac{14x^3-65x^2+55x-10}{(2x^3-3x^2+1)(x^2+7x-6)}$$

of which the two factors in the denominator have no factor in common (i.e. are *prime* to each other). This can be shown to split up into two proper fractions *in one way only*.

For suppose it equal to

$$\frac{ax^2+bx+c}{2x^3-3x^2+1}+\frac{dx+e}{x^2+7x-6}.$$

By clearing fractions we get the identity

$$0x^4+14x^3-65x^2+55x-10$$
$$\equiv(ax^2+bx+c)(x^2+7x-6)+(dx+e)(2x^3-3x^2+1).$$

[The $0x^4$ on the left is put in, because there is a term in x^4 on the right.]

Equating coefficients we get:

$$\begin{array}{lr}
(x^4) & a+2d \qquad\qquad = 0, \\
(x^3) & 7a+b-3d+2e = 14, \\
(x^2) & -6a+7b+c-3e = -65, \\
(x) & -6b+7c+d \qquad = 55, \\
\text{(constant term)} & -6c+e \qquad = -10.
\end{array}$$

These 5 linear equations determine *uniquely* the 5 unknowns, a, b, c, d, e (unless they give no solution).

The solution will be found to be $a=2$, $b=-7$, $c=2$, $d=-1$, $e=2$.

Thus the given fraction can be split up into the suggested fractions in *one and only one* way.

This is an example of the following *general theorem.*

If P, Q are polynomials with no common factor and of degrees p, q, respectively, and if R is a polynomial of degree r such that $r<p+q$, then the proper fraction $\dfrac{R}{P\,.\,Q}$ can be expressed *uniquely* as the sum of two *proper* fractions $\dfrac{S}{P}+\dfrac{T}{Q}$.

For multiplying up gives $R=S\,.\,Q+P\,.\,T$, of which each side is of degree $p+q-1$ or less, and equating coefficients (some of which may be zero) we get $(p+q)$ linear equations which will determine *uniquely* the p coefficients of S and the q coefficients of T.

Examples 54

1. (i) In $\dfrac{x}{(x+2)(x^2+4)}$ find the partial fraction with $(x+2)$ as denominator.

   (ii) If $\dfrac{x+5}{(x-1)(x^2+x+6)}$ is put into partial fractions $\dfrac{a}{x-1}+\ldots$, find a.

2. Express $\dfrac{x^2}{(x-a)(x-b)(x-c)}$ and $\dfrac{x^2}{(x+a)(x+b)(x+c)}$ in partial fractions.

3. Find partial fractions for $\dfrac{1}{(x-1)(x-2)(x-3)(x+5)}$.

4. Express $\dfrac{2x^3-17x^2+51x-70}{(x+2)(x-2)^3}$ in partial fractions.

5. Give the partial fractions for $\dfrac{1}{(x+1)^3(x-1)}$.

6. Express in partial fractions $\dfrac{63-49x+9x^2}{x(3-x)^2}$.

7. If $\dfrac{2}{(x^2+1)(x-1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$ find A, B, C.

8. If $\dfrac{2x^3-3x^2+5x+1}{x^3+1}=A+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2-x+1}$ find A, B, C, D.

9. If $y=1+x$ show that $\dfrac{1+2x}{(1+x)^3(1-2x)}=\dfrac{2y-1}{y^3(3-2y)}$.

   Express the R.H.S. in the form $\dfrac{A}{3-2y}+\dfrac{B+Cy+Dy^2}{y^3}$.

   Hence find the partial fractions for the L.H.S.

10. Find a, b, c if $x^3 \equiv (x-2)^3 + a(x-2)^2 + b(x-2) + c$.

11. Express $3x^4 - x^3 + 5$ as a function of x' where $x' = x + 3$.

12. Express $2x^3 + 5x^2 + 7x + 3$ as a function of y if $y = x - 1$ and also as a function of z if $z = x + 1$.

Check this last answer by using the change $y = z - 2$.

13. Determine the values of the constants A, B, C, D if

$$\frac{x^2+1}{(x^2-1)(x^2+x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1}$$

for all values of x. (L.)

14. Find the values of A, B, and C that the identity

$$\frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-x} \equiv \frac{2x}{(1+x)^2(1-x)}$$

may be true. (L.)

15. Express $\dfrac{4x^3}{(x-1)^2(x-2)^2(x-3)}$ as the sum of partial fractions.

(O. & C.)

[In Nos. 16, 17, the function will be found as the sum of four functions which need not be added together.]

16. Find the cubic function of x which takes the values A, B, C, D when $x = a, b, c, d$ respectively.

17. Find the cubic function of x which takes the values 7, 9, 5, 2 when $x = -1, 0, 1, 2$ respectively.

18. Find partial fractions for $\dfrac{x^2}{(x-1)^3(x^2+1)}$.

19. Find partial fractions for $\dfrac{(5+x)^2}{(1-x)^2(2+x)^4}$.

[If $2 + x = y$, prove by division in ascending powers of y, using $(1-x)^2 = (3-y)^2$, that the fraction equals

$$\{1 + \tfrac{4}{3}y + \tfrac{8}{9}y^2 + \tfrac{4}{9}y^3 + (\tfrac{16}{9}y^4 - \tfrac{4}{9}y^5)/(3-y)^2\} \div y^4,$$

and deduce the expression in partial fractions in terms of x.]

20. If $\dfrac{x^4 - 2x^2}{(x^3+3x-1)(x^2-5x-3)} = \dfrac{ax^2+bx+c}{x^3+3x-1} + \dfrac{dx+e}{x^2-5x-3}$,

find 5 equations to determine a, b, c, d, e and show that a, b, c are given by the equations

$$\left.\begin{aligned} 5a - b + 3c &= 0 \\ 6a + 5b - c &= 5 \\ a - 3b - 14c &= 1 \end{aligned}\right\}.$$

Change of Number-Scale

An interesting application of the method of successive division for change of variable is in changing the "*base*" of the scale in which a number is expressed.

In the usual decimal scale with ten as the base (or *radix*),

$$2416 \equiv 2 \times 10^3 + 4 \times 10^2 + 1 \times 10 + 6.$$

Example I. Suppose it is required to express this number in the scale of 7, i.e. as $a \times 7^4 + b \times 7^3 + c \times 7^2 + d \times 7 + e$.

We see that if this last number is divided by 7 the quotient is

$$a \times 7^3 + b \times 7^2 + c \times 7 + d$$

and the remainder is e, and that if we repeat the dividing by 7 we get first d as remainder, and then c, and so on.

Here the process of division is simple arithmetic.

The work at the side shows that

$$2416 = 1 \times 7^4 + 0 \times 7^3 + 0 \times 7^2 + 2 \times 7 + 1.$$

The right-hand side would be written as 10021 in the scale of 7.

$$\begin{array}{rl} 7\,)\,\underline{2416} & \\ 7\,)\,\underline{345} & \ldots 1 \\ 7\,)\,\underline{49} & \ldots 2 \\ 7\,)\,\underline{7} & \ldots 0 \\ 1 & \ldots 0 \end{array}$$

Example II. Express 38214 in the scale of 5.

$$\begin{array}{rl} 5\,)\,\underline{38214} & \\ 5\,)\,\underline{7642} & \ldots 4 \\ 5\,)\,\underline{1528} & \ldots 2 \\ 5\,)\,\underline{305} & \ldots 3 \\ 5\,)\,\underline{61} & \ldots 0 \\ 5\,)\,\underline{12} & \ldots 1 \\ 2 & \ldots 2 \end{array}$$

38214 in scale of $10 = 2210324$ in scale of 5.

The reverse processes are more difficult, for they involve working in the scales of 7 and 5 respectively.

Remember that in the scale of 7 the figures 10 mean seven and not ten.

Example III. If 10021 is a number in scale 7 and 2210324 one in scale 5, express each in the scale of ten.

Working in the scale of 7 and dividing by ten (t) [which would be 13 in scale of 7]:

$$\begin{array}{rl} t\,)\,\underline{10021} & \\ t\,)\,\underline{463} & \ldots 6 \\ t\,)\,\underline{33} & \ldots 1 \\ 2 & \ldots 4 \end{array}$$

t into 10 won't go, into 100 or $49 + 0$ goes 4 times and nine over ; into nine $\times 7 + 2$ ...

t into $3 \times 7 + 3$ goes 2 and 4 over.

Result : 2416 in scale of ten.

Working in scale of 5 and dividing by ten :

$$\begin{array}{r}
t\,)\,\underline{2210324} \\
t\,)\,\underline{110241}\;\ldots\;4 \\
t\,)\,\underline{3012}\;\ldots\;1 \\
t\,)\,\underline{123}\;\ldots\;2 \\
3\;\ldots\;8
\end{array}$$

We start $2 \times 5 + 2 =$ twelve ; ten goes once and 2 over.

Result : 38214 in scale of ten.

Conversion of Decimal and other Radix Fractions

Successive multiplication instead of successive division is needed for conversion of decimals into radix fractions in other scales and vice versa.

Example I. Express the decimal ·756 in the scale of 8 as $\frac{a}{8} + \frac{b}{8^2} + \frac{c}{8^3} + \ldots$.

Multiplication by 8 will give a as the integral part : a second multiplication by 8 of the fractional part will give b, and so on.

The work shows that the new radix fraction is ·6030 … . It will not terminate because ·756 cannot be converted into a whole number by multiplication by a power of 8.

$$\begin{array}{r}
\cdot 756 \\
\underline{8} \\
6\,)\,\cdot 048 \\
\underline{8} \\
0\,)\,\cdot 384 \\
\underline{8} \\
3\,)\,\cdot 072 \\
\underline{8} \\
0\,)\,\cdot 576
\end{array}$$

Example II. Convert ·60304 in scale 8 into a decimal fraction.

Repeated multiplication by ten, *in the scale of* 8, is used.

The first four figures are ·7556.

This agrees to three figures with ·756.

$$\begin{array}{r}
\cdot 60304 \\
\underline{t} \\
7\,)\,\cdot 43650 \\
\underline{t} \\
5\,)\,\cdot 44220 \\
\underline{t} \\
5\,)\,\cdot 52640 \\
\underline{t} \\
6\,)\,\cdot 54100
\end{array}$$

Examples 55

1. Express the numbers 7162 and 5555 which are in the scale of 10 :

(a) in the scale of 8 ; (b) in the scale of 4.

Check your answers by reconverting them into the scale of 10.

2. Express 79 in the scale of 2 and show that 103 is equal to 1,100,111 in the scale of 2.

[Since any number can be expressed in the scale of 2, this example shows that any weight which is a whole number of pounds may be balanced if we have available *one each* of the weights 1 lb., 2 lb., 4 lb., 8 lb., 16 lb., 32 lb., etc. That 103 is equal to 1,100,111 in the scale of two shows that 103 lb. can be balanced using the weights 1 lb., 2 lb., 4 lb., 32 lb. and 64 lb.]

3. If the weights 1 lb., 2 lb., 4 lb., ... , up to 128 lb., are available, what is the greatest weight which can be balanced and which of the weights must be used to balance 165 lb.?

4. Express 95 and 137 in the scale of 3.

5. Show that if -1, written $\bar{1}$, is allowed as a digit, the numbers 2102 and 12012 in the scale of 3 may be written as $1\bar{1}11\bar{1}$ and $1\bar{1}\bar{1}1\bar{1}\bar{1}$ respectively in the same scale.

[Since 65 in the scale of ten $=$ 2102 in the scale of 3, this result shows that if the weights used may be placed in either pan of the balance, 65 lb. with 1 lb. and 27 lb. in same pan are balanced by 3 lb., 9 lb. and 81 lb. in the other.]

6. Prove that Example 5 shows how to balance 140 lb. if the weights available are one each of 1 lb., 3 lb., 9 lb., 27 lb., 81 lb., and 243 lb. (which may be used on either side of the balance).

What is the greatest weight which can be balanced if this set of weights is available? Show how to use the weights to balance half this maximum.

[If weights may be placed in both pans of the balance, this set—1 lb., 3 lb., 9 lb., 27 lb., etc.—gives the *least* number of (standard) weights needed for weighing. If the (standard) weights may only be put in one pan, the previous set, 1 lb., 2 lb., 4 lb., 8 lb., etc., gives the least number.]

7. Express the numbers 478 and 3443 of the usual scale each in the scale of twelve, taking t and e to stand for the extra digits (ten and eleven) needed in the scale of 12.

[The scale of twelve has the advantage over the scale of ten, that its base twelve can be quartered as well as halved.]

8. Show that $2t^3+4t^2+t+6=2(t-3)^3+22(t-3)^2+79(t-3)+99$. In this put $t=$ ten and explain how to reduce the R.H.S. to the number 10021 in scale of 7.

Also use the result to convert 2416 in scale 12 into a number in the scale of 9. Verify this by successive division by 9 working in the scale of 12.

9. (i) Express $\cdot 5672$, scale of ten, as a radix fraction in scale of 5, and check by converting back. (ii) Express $\cdot 6875$ in scale of 2.

10. $\cdot 755$ is a radix fraction in scale of 8: convert it into a radix fraction in scale of 5 to six places. Also express it in scale of 10 exactly.

11. Prove $25\cdot 248$ in scale of $10 = 100\cdot 111$ in scale of 5.

One-to-one Correspondence

If two variables x and y are connected by an equation in such a way that any value of x gives *one and only one* value of y and also that any value of y gives *one and only one* value of x, then the equation must be of the form

$$pxy - qx - ry + s = 0 \qquad \text{(I)}$$

since if x is given, it must be linear in y and vice versa.

It is essential also that $ps \neq qr$, for if $ps = qr$ then the equation may be written as $\frac{1}{p}(p^2xy - pqx - pry + qr) = 0$,

i.e. as $\frac{1}{p}(px - r)(py - q) = 0$, which makes $x = \frac{r}{p}$ or $y = \frac{q}{p}$.

x and y connected by an equation such as (I) may be used to fix pairs of corresponding points on a line, as shown in Fig. 36.

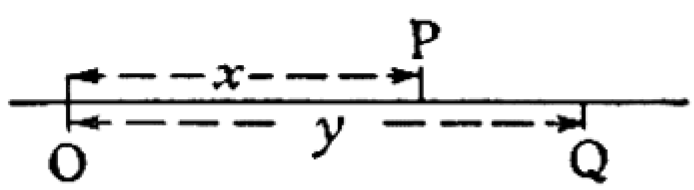

FIG. 36

Again, x and y may be taken as the coordinates (x, y) of a point referred to perpendicular axes. In this case, if $p = 0$ the equation reduces to $qx + ry = s$, which is the equation of a straight line, and the change is made by a substitution of the form $x = ay + b$; or if $p \neq 0$ we can divide through by p and obtain an equation of the form

$$xy - ax - by + c = 0$$

or

$$(x - b)(y - a) = ab - c. \qquad \text{(II)}$$

The graph of this is a rectangular hyperbola (Fig. 37) which shows the case when a, b, and $ab - c$ are each positive.

The substitutions for change of variables are

$$y = \frac{ax - c}{x - b} \quad \text{and} \quad x = \frac{by - c}{y - a},$$

and the asymptotes of the hyperbola are $x = b$ and $y = a$.

There are two points where the values of x and y are equal. These are L and M in the diagram, being the points where the curve is met by $y=x$; they are given by $x^2-(a+b)x+c=0$ and $x=y$.

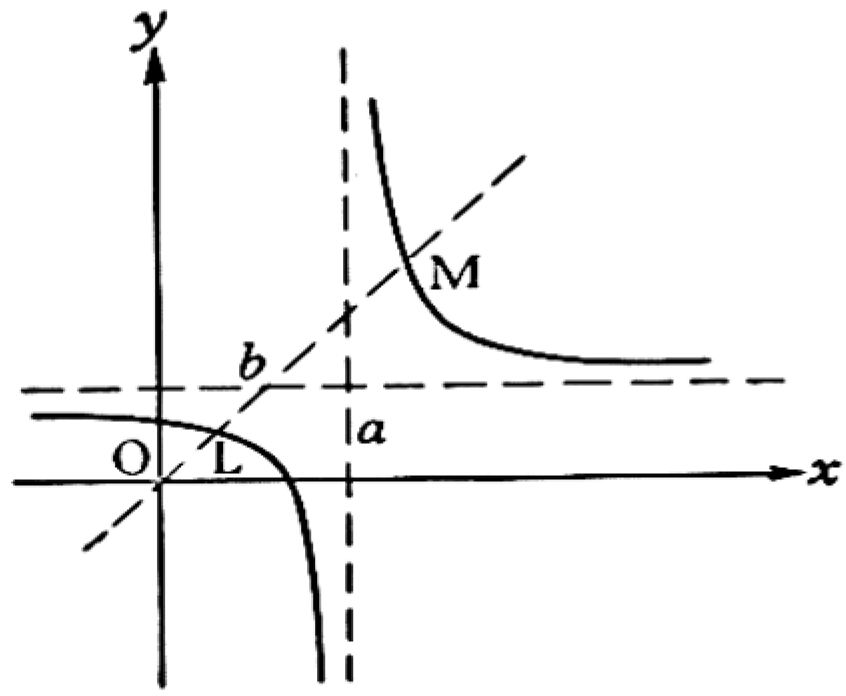

FIG. 37

Example. If x_1, x_2 give values y_1, y_2 in equation (II), express x_1-x_2 in terms of y_1 and y_2.

Solution.

$$x_1-x_2=\frac{by_1-c}{y_1-a}-\frac{by_2-c}{y_2-a}$$
$$=\frac{(by_1y_2-cy_2-aby_1+ac)-(by_1y_2-cy_1-aby_2+ac)}{(y_1-a)(y_2-a)}$$
$$=\frac{-(ab-c)(y_1-y_2)}{(y_1-a)(y_2-a)}.$$

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1. If $x=ay+b$ and values x_1, x_2 give values y_1, y_2 show that x_1-x_2 is a multiple of y_1-y_2, and that x_1-2x_2 is not a multiple of y_1-2y_2.
2. If $ax+by=c$ express x^2-px+q as a function of y.
3. If $xy-5x-7y+27=0$ find the values for which $x=y$.
4. Given $xy+4x-9y=0$ find the values of x and y for which (i) $x=y$; (ii) $x=2y$.
5. Using the result of the worked Example I show that if the equation $xy-ax-by+c=0$ gives y_1, y_2, y_3, y_4 as the four values corresponding to x_1, x_2, x_3, x_4, then
$$\frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)}=\frac{(y_1-y_2)(y_3-y_4)}{(y_1-y_4)(y_3-y_2)}.$$

[This result is of great importance in algebraic geometry.]

6. If $pxy - q(x+y) + s = 0$ is the equation determining corresponding points on a line as in Fig. 36, show that there are real self-corresponding points (when $x = y$) if $q^2 > ps$.

Also show that the same corresponding point-pairs may be obtained by the equation $x'y' = (q^2 - ps)/p^2$ when $x' = x - q/p$ and $y' = y - q/p$.

Miscellaneous Examples 57

1. Add together $\dfrac{1}{x} + \dfrac{1}{x+5} + \dfrac{1}{x+10} + \dfrac{1}{x+15}$.

   Check by putting $x = 10$.

2. Put into partial fractions:

   (i) $\dfrac{x^2}{(x+1)(x+2)(x+3)}$; (ii) $\dfrac{x^3}{(x+1)(x+2)(x+3)}$.

3. Find the partial fractions for

   (i) $\dfrac{5x+2}{x^3+1}$; (ii) $\dfrac{5x+2}{(x+1)^3}$.

4. Find the partial fractions for $\dfrac{x^2}{(1+ax)(1+bx)(1+cx)}$.

5. If $n^4 \equiv n(n-1)(n-2)(n-3) + an(n-1)(n-2) + bn(n-1) + cn + d$, find a, b, c, d.

6. Express $3x^3 + 2x^2 - 7x + 5$ as a function of y where $y = x - 1$.

7. Express $x^4 + 12$ as a function of y where (i) $y = x - 3$; (ii) $y = x + 3$.

8. (i) Express 2718 in the scale of 4.
   (ii) If $\cdot 2\dot{3}$ is a radix fraction in the scale of 4 express it as a decimal.

9. (i) Show how to weigh 100 lb. with weights of 1 lb., 2 lb., 4 lb., 8 lb., 32 lb. and 64 lb.
   (ii) Show also how to weigh 100 lb. with weights of 1 lb., 3 lb., 9 lb., 27 lb. and 81 lb., which may be placed in either pan of the balance.

10. Express the decimal of $\cdot\dot{3}$ as a radix fraction in the scale of 7; also the decimal $\cdot\dot{5}$; *deduce* the value of the decimal $\cdot\dot{2}$ as a radix fraction in the scale of 7 and check by direct calculation.

11. A one-one relation of the form $xy - ax - by + c = 0$ exists between the numbers x and y. When x has the values 5, 6, 9 the corresponding values for y are 17, 12, 7 respectively. Find the value of y corresponding to $x = 13$, by discovering first the one-one relation.

    Also find the self-corresponding values.

Test Papers A

A.I

1. Some soldiers are conveyed from A to B by x lorries each carrying 20 men, assisted by $x-7$ planes each of which carries 6 men and makes the trip 3 times. If the total number of soldiers conveyed is 824, find x. (B.)

2. Prove that
$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
and use this result to factorise
$$x^3+8y^3+27z^3-18xyz. \qquad \text{(B.)}$$

3. Solve the equation $x^2-x-5=0$ giving the answers (a) as surds, (b) as decimals to 3 significant figures.
Find the square of the positive answer, preferably using the form given in (a), and prove that it satisfies the equation $x^2-11x+25=0$. (B.)

4. Find values of a, b and c independent of x, so that $2-3x+5x^2$ may be identically equal to $a-b(c-x)^2$. How does your answer show the minimum or maximum value of the expression?
For what range of values of x is the gradient of the graph of this expression less than seven?

5. Taking 1 in. as unit on each axis draw the graph of $y=\frac{1}{x}$ for values of x from $\frac{1}{2}$ to 4.
On the same axes and with the same scales draw the graph of $x=4-2y$ and read off the values of x at the intersections of the two graphs.
Check these values by solving an equation. (B.)

6. If $x+y=u$ and $xy=v$, prove that $x^2+y^2=u^2-2v$ and that
$$x^3+y^3=u^3-3uv.$$
Using these substitutions solve for u and v the equations
$$x^3+y^3=13(x+y),$$
$$x^2-2xy+y^2=x+y-6.$$
Why must $u=0$ be rejected?
[The values of x and y need not be found.] (L.)

7. (i) Given that $\log 2=\cdot 3010$ and $\log 3=\cdot 4771$ find without reference to the tables the logarithms of: $\sqrt{2}$, 5, 162, 27, $\sqrt[3]{\frac{1}{9}}$, making it clear how the results are obtained.
(ii) Solve the equation
$$\log_{10}(5x+6)=2\log_{10}(5x-6). \qquad \text{(L.)}$$

A.II

1. (*a*) Factorise:

$$4x^2 - 10x + 6 \quad \text{and} \quad x^2 - 5x + 6 - a(x^2 - 4).$$

(*b*) State the Remainder Theorem in the form applicable to factorisation. Hence or otherwise factorise

$$x^3 - 19x + 30.$$ (B.)

2. (*a*) Solve $\dfrac{8x-2}{5} - \dfrac{3x+1}{7} = 10.$

(*b*) Solve $\dfrac{30}{x} + \dfrac{44}{y} = 10$; $\dfrac{40}{x} + \dfrac{55}{y} = 13.$

(*c*) Simplify:

$$\frac{1}{x^2 - 3x + 2} - \frac{2}{x^2 - 5x + 6} + \frac{3}{x^2 - 4x + 3}.$$ (B.)

3. (*a*) Multiply $x^2 - 1 + \dfrac{1}{x}$ by $x^2 + 1 + \dfrac{1}{x}$ and check your result by putting $x = 3$.

(*b*) Verify that if $x = \dfrac{2(a^2 + b^2)}{a + b}$ then $\dfrac{x-a}{b} + \dfrac{x-b}{a} = \dfrac{a^2 + b^2}{ab}$. (B.)

4. (*a*) Solve $3x - 4y = 5$; $4x^2 - 3xy = 27$.

(*b*) Apples are graded by size into two sorts, the first x to the lb., the second $x + 1$ to the lb. 40 of the smaller weigh $\frac{1}{2}$ lb. more than 30 of the larger. Find x. (B.)

5. Draw the graphs of $y = (x + 2)(2x - 5)$ and of $y = \dfrac{8x}{3}$ on the same axes from $x = -3$ to $x = 4$.

[Take 1 in. as the unit for x, 0·2 in. as the unit for y.]

Find their points of intersection and verify by forming and solving the quadratic equation which gives them.

Find also the values of x and y at the lowest point of the former graph. (B.)

6. (*a*) Put into logarithmic form the equation $2^x = 3^5 \div 5^3$ and find the approximate value of x.

(*b*) If $x = \sqrt{5} - \sqrt{3}$, show that $x^3 = 14\sqrt{5} - 18\sqrt{3}$ and verify approximately. (B.)

7. For what range of values of x is $x^2 - 4x$ negative?

If $y = |x^2 - 4x|$ for what range of values of y do 4 values of x correspond to each value of y?

Find the 4 values of x if $y = 3$.

[A solution from a graph will be accepted.]

A.III

1. Simplify

$$\frac{a}{x+\frac{1}{2}}+\frac{a}{x-\frac{1}{2}}-\frac{2a}{x}.$$

Thinking of a as a distance in miles and x as a speed in miles per hour, interpret your result as giving the time by which a current retards a boat on a journey to a certain distance and back again, giving a numerical result for the case when $a=10$, and $x=2\frac{1}{2}$.

2. (i) For what value of a is x^2+x+1 a factor of the expression

$$x^3+5x-9-(x-a)(x+7)?$$

Find also the other factor.

(ii) Factorise $a^2-2aby-b^2\left(x^2-xy-\frac{3y^2}{4}\right)$.

3. Solve the equations:

$$\text{(i) } x^2-13x-378=0.$$

$$\text{(ii) } x+2y=7;\quad x^2+2y^2=17.$$

$$\text{(iii) } x-y=1;\quad y^2-7x=1;\quad x-z^2=0.$$

4. Simplify

$$|2-x|+|x-3|+|4-x|.$$

(i) If $2<x<3$. (ii) If $3<x<4$.

5. A quadratic function x^2-x+2 is selected. By writing it as (a square containing $x+$a constant), find its minimum value.

Draw its graph for a range of 2 on each side of its minimum value. Use the graph to solve the equation $x^2-x-1=0$.

6. (i) Find the value of $\log 32 \div \log 16$.

Do this, if possible, without use of tables. Extra credit may be obtained by taking 3 as the logarithmic base instead of 10.

(ii) Evaluate £$400\times(1{\cdot}055)^7$, and £$400\times(1{\cdot}055)^{-7}$ as accurately as your logarithm tables permit. State your results in the language of compound interest.

7. Show that the equation $x^4+12x-5=0$ has a root a little greater than ·4 and find the second decimal place.

A.IV

1. (i) Simplify $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x-1+x^{-1})$.

(ii) Express $2^{\frac{1}{3}}$ as a power of 10 and $10^{\frac{1}{3}}$ as a power of 2.

[Answers to be given to 4 figures.] (L.)

2. (i) Factorise (a) $3x^3 - 81$, (b) $x^4 + 4a^2x^2 + 16a^4$.
(ii) What is the value of a if $2x^3 + 9x^2 + ax - 6$ is divisible by $2x + 3$ without remainder? (L.)

3. Solve the equations
(i) $\sqrt{x^2 - 5x} + \sqrt{x^2 - 5x + 12} = 6$.
[Hint. Put $x^2 - 5x = y$.]
(ii) $2x^2 + xy - y^2 = 5$, $2x - 3y + 5 = 0$. (L.)

4. (i) Simplify $\dfrac{1}{2x^2 + 18x} + \dfrac{1}{2x^2 - 18x} - \dfrac{1}{x^2 - 81}$.
(ii) If $a - p + \dfrac{2}{a}$ is the square root of $a^2 - 6a + 13 - \dfrac{12}{a} + \dfrac{4}{a^2}$ find p. (L.)

5. (i) Without using tables simplify
$$1 + 3 \log_{10} 2 + \log_{10} 5^2 + \log_{10} \tfrac{1}{2}.$$
(ii) Prove that $\log(n + 1) + \log(n - 1) < 2 \log n$, and that, if $\log 1001 = 3{\cdot}00043$, then, for any value of $n > 1000$,
$$\log(n + 1) - \log n < {\cdot}0005.$$ (L.)

6. Draw the graphs $y = x^2 - x$ and $y = \dfrac{2 - x}{x}$ and use them to find the real root of the equation $x^3 - x^2 + x = 2$. (L.)

7. If α, β are the roots of $px^2 + qx + r = 0$, find the values of $\alpha + \beta$ and $\alpha\beta$ in terms of p, q and r.
α, β are the roots of $ax^2 - x + b = 0$; α', $-\beta$ are those of $a'x^2 - x + b' = 0$; α, α' those of $Ax^2 - x + B = 0$.

Prove (i) $\dfrac{1}{a} + \dfrac{1}{a'} = \dfrac{1}{A}$; (ii) $\dfrac{1}{b} + \dfrac{1}{b'} = \dfrac{1}{B}$. (B.)

A.V

1. (a) Resolve into factors:
(i) $p^3 + 4p^2 - 4p - 16$; (ii) $(p^2 + 2p)^2 - 7(p^2 + 2p) - 8$.
(b) Find the values of p and q if $x^3 + px^2 - 10x + q$ is exactly divisible by $x^2 - x - 6$. In this case find the remaining factor of the expression. (L.)

2. Solve the following equations:
(a) $\sqrt{4 + x} + \sqrt{6 - x} = 2\sqrt{x - 1}$;
(b) $\left.\begin{aligned} x^2 + 2xy &= 5x + 10y \\ x + y &= 7 \end{aligned}\right\}$. (L.)

3. (i) If $\log_a x + \log_a x^2 + \log_a x^3 + \log_a x^4 = 5$ prove that $x = \sqrt{a}$.
(ii) Simplify $\dfrac{\log 16 \times \log 27}{\log 9 \times \log 64}$.
(iii) If $\log(x - y) = \frac{1}{2}(\log x + \log y)$ prove that
$$2 \log(x + y) = \log 5 + \log x + \log y.$$ (L.)

4. If
$$\frac{x}{\begin{vmatrix} 5 & 7 \\ -2 & 4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 7 & 3 \\ 4 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & 5 \\ 1 & -2 \end{vmatrix}}$$
show that $3x + 5y + 7 + k(x - 2y + 4) = 0$ for all values of k.

5. The electrical resistance of a wire, in ohms, varies directly as its length and inversely as the square of its diameter.

A copper wire of diameter 0·116 in. and length 440 yds. has a resistance of 1 ohm. Find, to three significant figures, the resistance of a piece of copper wire of diameter 0·025 in. and length 1,000 yds. (B.)

6. Express $\dfrac{2x+3}{x(x+1)(x+2)}$ in partial fractions.

Prove that, for *large* values of x, the given expression is approximately equal to $\dfrac{2}{x^2}$. (O. & C.)

7. If α, β, γ are the roots of $x^3 + 3ax^2 + b = 0$, write down the values of $\Sigma\alpha$, $\Sigma\beta\gamma$ and $\alpha\beta\gamma$.

Show that $\alpha^3 + \beta^3 + \gamma^3 = -27a^3 - 3b$.

A.VI

1. Factorise as the difference of two squares:
   (ii) $a^4 + b^4 + c^4 - 2b^2c^2 + 2c^2a^2 + 2a^2b^2$;
   (ii) $a^4 + b^4$. (B.)

2. State the meaning given to $a^{-\frac{p}{q}}$ where p and q are positive integers and explain why it is given this meaning.

Put in order of ascending magnitude, $8^{-\frac{2}{3}}$, $3^{-\frac{6}{8}}$, $5^{-\frac{2}{3}}$ giving your reasons clearly. (B.)

3. State the remainder when a polynomial in x, denoted by $f(x)$, is divided by $x - a$ and also the remainder when it is divided by $x + a$, and prove your statements.

If the polynomial is $x^4 + px^3 + qx^2 + r$ and the two remainders are equal, what do you know about any of the coefficients p, q, r? (B.)

4. For what values of x is $\dfrac{(x+5)(x-2)}{(x+3)(x-4)}$ zero and for what values is its reciprocal zero? Describe its changes of sign as x changes from -10 to $+10$. For what values of x is it (i) equal to 2, (ii) equal to 1? (B.)

5. (i) Factorise $x^3 + 2x^2 - x - 2$.

   (ii) Express $\dfrac{7x^2 + 23x + 6}{2(x^3 + 2x^2 - x - 2)}$ in partial fractions.

6. Part of the total expenses of running a trip are constant, while the other part varies directly as the number of people taken and inversely as the

speed on the journey. For a party of 100 at a speed of 40 m.p.h. the cost is £48, while for a party of 200 at a speed of 60 m.p.h. the cost is £63. Find the cost for a party of 500 at a speed of 75 m.p.h. (B.)

7. Show that the equation

$$x^3 - 12x + 8 = 0$$

has three real roots, one greater than unity, another positive and less than unity, and the third negative, the sum of the three roots being zero.

Find the value of the root between 0 and 1 correct to two decimal places. (L.)

A.VII

1. Solve the equations:

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{x}=\frac{1}{a+b+x}$, $a+b \neq 0$.

(ii) $\sqrt{(x-5)}+\sqrt{(18-x)}=\sqrt{(x+16)}$. (O. & C.)

2. Trace the changes of sign of the expression $2-x-x^2$ as x varies from $-\infty$ to $+\infty$ by marking segments on the x-axis with plus and minus signs, indicating clearly the values of x at the end points of the segments.

Trace the changes of sign of the expression $(x^2-3x+2)/(2x+1)$ in a similar manner. (O. & C.)

3. If α and β are the roots of the equation $x^2-px+q=0$, form the equation whose roots are $\alpha+1/\beta$, $\beta+1/\alpha$.

Denoting the roots of your new equation by α' and β', form the equation whose roots are $1/\alpha'$, $1/\beta'$, and show that the roots of this equation are proportional to those of the original equation. (O. & C.)

4. Sketch the graphs of: (i) $y = |x+y| + |x+3|$;

(ii) $|y^2-4| = |x^2-4|$.

5. (*a*) Without using logarithm tables show that

$$\log 27 + \log 16 - \log 9 - \log 12 = 2 \log 2.$$

(*b*) Find the value of $\log_a 32 \div \log_a 4$.

6. The two equations,

$$18x^2+17x-15=0, \quad 9x^3+31x^2-83x+35=0,$$

have a common root; find it and verify your answer by substitution in each equation.

Find p and q so that this common root may be a repeated root of the equation $81x^3+px^2+qx-25=0$. (O. & C.)

7. If $f(x)=x^3-x^2-3x+3$ find the value of $f(1{\cdot}7)$ and $f(1{\cdot}8)$. Use Newton's method to show that $1{\cdot}73$ is an approximate root of $f(x)=0$.

Also solve the equation by factorising $f(x)$.

A.VIII

1. (i) Solve the simultaneous equations

$$2x+3y+1=0, \quad 4x^2-2xy+y^2+4x=11.$$

(ii) Factorise completely

$$a(b-c)^3+b(c-a)^3+c(a-b)^3. \qquad \text{(O. \& C.)}$$

2. The roots of the equation $ax^2+bx+c=0$ are α and β. Without solving the equation, prove that

$$\alpha+\beta=-\frac{b}{a}, \quad \alpha\beta=\frac{c}{a}.$$

Given that α is equal to $p\beta$, prove that

$$b^2p=ac(1+p)^2.$$

Find the equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$, where α and β are the roots of the equation

$$3x^2-2x+1=0. \qquad \text{(O. \& C.)}$$

3. If $y=ax(x-1)(x-2)+b(x+1)(x-1)(x-2)$
$$+c(x+1)x(x-2)+d(x+1)x(x-1)$$
and if corresponding values of x and y are given by the table:

x	-1	0	1	2
y	36	8	-4	15

find the values of a, b, c, d and hence the value of y when $x=\frac{1}{2}$. (L.)

4. When a wire is stretched between two posts, the sag, d feet, of the wire varies directly as the square of its length, l feet, and the weight, w lb., of the wire per foot, and inversely as the tension T lb. in the wire. For a wire 100 ft. long weighing ·064 lb. per foot at a tension of 240 lb., the sag was 4 inches.

Find a formula for d in terms of l, w and T.

5. Express as the sum of partial fractions:

(i) $\dfrac{x}{(x-1)(x-2)}$; (ii) $\dfrac{x^2}{(x-1)^2(x-2)}$; (iii) $\dfrac{x^3}{(x-1)^3}$. (O. & C.)

6. With the same axes and the same scales draw the graphs of the functions

$$\frac{x^2-5x+4}{x^2+4} \text{ and } \tfrac{1}{2}(2-x)$$

for values of x between -4 and $+4$.

Prove (do not merely verify) that the abscissae of the three points of intersection of the graphs satisfy the equation $x^3-6x=0$, and deduce from your graphs the square root of 6. (O. & C.)

7. Convert 31459 into a number in the scale of seven.

Working in the scale of seven convert your answer into the scale of 5, and check by converting 31459 directly into the scale of 5.

CHAPTER VII

SEQUENCES; SERIES; THE PROGRESSIONS; MEANS

Sequences and Series

A surprisingly large part of advanced mathematics is concerned with sequences and series.

A *sequence* consists of a set of numbers or *terms* following each other in a definite order, such that each one can be calculated in a definite way from a knowledge of its position in the order or from a knowledge of the previous terms.

The natural numbers 1, 2, 3, 4, 5, ... ; the even numbers 2, 4, 6, 8, ... ; the powers $-2, 4, -8, 16, -32, \ldots$ are instances.

Any of these might in ordinary language be called a series of numbers, but in mathematics a distinction is made between a *series* and a *sequence*. The terms of a *series* are connected with + or − signs, as in $1+2+3+4+5+\ldots$ or $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$, while in the case of a *sequence* there is no suggestion of the terms being added together, and they are written down with commas between them as in the case 1, 3, 6, 10, 15, ...

The type of sequence given by the sums of 1, 2, 3, ... terms of a series is especially important; the example just given shows the sums of 1, 2, 3, ... terms of the series of natural numbers.

More exactly, if n is any positive integer and a rule is given so that a definite number u_n is determined corresponding to n, then the numbers $u_1, u_2, u_3, u_4, \ldots$ form a *sequence*.

If we write $u_1+u_2+u_3+u_4+\ldots+u_n$ we have a finite *series* of n terms, and if we take $S_1=u_1$, $S_2=u_1+u_2$, $S_3=u_1+u_2+u_3$, ... and $S_n=u_1+u_2+u_3+\ldots+u_n$, we have a new *sequence* $S_1, S_2, S_3, \ldots$ consisting of the sums of 1, 2, 3, ... terms of the series.

Thus any series depends on the sequence of its terms and provides a fresh sequence of its sums.

Note that by definition a sequence is unending, just as the sequence of positive integers is unending. A series may be finite if we decide to stop at a particular term, or may be carried on indefinitely, in which case it is called an "infinite" series.

Examples 58

1. If the following sequences continue in the way suggested by the first few terms, find in each case, the 20th term and the nth term.

   (i) $1, 3, 5, 7, \dots$; (ii) $1^2, 2^2, 3^2, 4^2, \dots$;
   (iii) $1, -2, 3, -4, \dots$; (iv) $1, -2, -5, -8, \dots$;
   (v) $2^2, 3^3, 4^4, 5^5, \dots$; (vi) $128, 64, 32, 16, \dots$.

2. Each term of a sequence (except the first two terms) is the sum of the two preceding terms ; the first two terms are 1 and 3 ; find the 10th term.

3. Two sequences have as their nth terms $3n+2$ and 3^n+2 ; write down the first four terms of each.

4. If the nth term of a sequence is $\left(1+\frac{1}{n^2}\right)^n$, show that its first term is greater than its third term by $\frac{458}{729}$.

5. If in a sequence $u_{2n-1}=1$ while $u_{2n}=-1$ for all integral values of n, write down the first five terms and show that the sum of any odd number of terms (from the start) is always the same.

6. In a sequence of increasing numbers, the differences between consecutive terms are 3, 2, 1, 3, 2, 1, 3, 2, 1, etc. The first three terms of the sequence are 3, 6, 8. Show that 20 and 30 are both terms of the sequence but 40 is not.

   Find the $(3n+2)$th term.

7. For a sequence whose law is the same as that in No. 2, the first two terms are 1 and 4 ; show that the 11th term exceeds the fifth term by 240.

8. If in a sequence each term is three times the previous one and the first term is 1, find the 9th term.

9. For the sequence whose nth term is $\frac{(-1)^n}{n+(-1)^n}$ write down four terms starting when $n=100$, and show that their sum is negative.

. In a sequence each term (except the first two) is the product of the two previous terms. The first two terms are 1 and 2. Which is the first term to be greater than 250?

The Progressions

The first types of series to be discussed, both in the history of mathematics and in the mathematical progress of the individual, are the three Progressions, the so-called Arithmetic, Geometric and Harmonic Progressions.

These are usually abbreviated into A.P., G.P., and H.P.

Arithmetic Progression (A.P.)

Here are some examples of A.P.'s ; each term is formed from the previous one by the addition of the same number ; and so the difference between any term and the previous one is a constant. This constant is called the *common difference.*

$$\left.\begin{array}{r} 3+\ 5+\ 7+\ 9+11+\dots \\ 22+27+32+37+42+\dots \\ 18+17+16+15+14+\dots \end{array}\right\}$$ The common differences here are 2, 5, -1 respectively.

In the series $22+27+32+37+42+\dots$

the 2nd term is $22+5$;
the 3rd term is $27+5$, i.e. $22+2\,.\,5$;
the 4th term is $32+5$, i.e. $22+3\,.\,5$; and so finally
the nth term is $22+(n-1)\,.\,5$, which in words can be stated " to get any term, add to the first term one less 5 than the number of the term ".

For the general A.P., where a is the *first term* and d is the *common difference*, the series is $a+(a+d)+(a+2d)+(a+3d)+\dots$, and the nth term is $a+(n-1)d$.

It should be noticed that though the algebraic expression

$$a+(n-1)d$$

has a value corresponding to every value of n, positive or negative, integral or fractional, rational or irrational, yet when considered as the nth term of an A.P. *the value of n must be a positive integer.* We are dealing here and in most of this chapter with *functions of a positive integral variable.*

The Sum of an A.P.

When a formula has been set down for the nth term of a series, the next question asked is whether a formula can be found for the sum of n terms. This can be done for an A.P.

Consider first the series of natural numbers $1+2+3+\dots+n$:

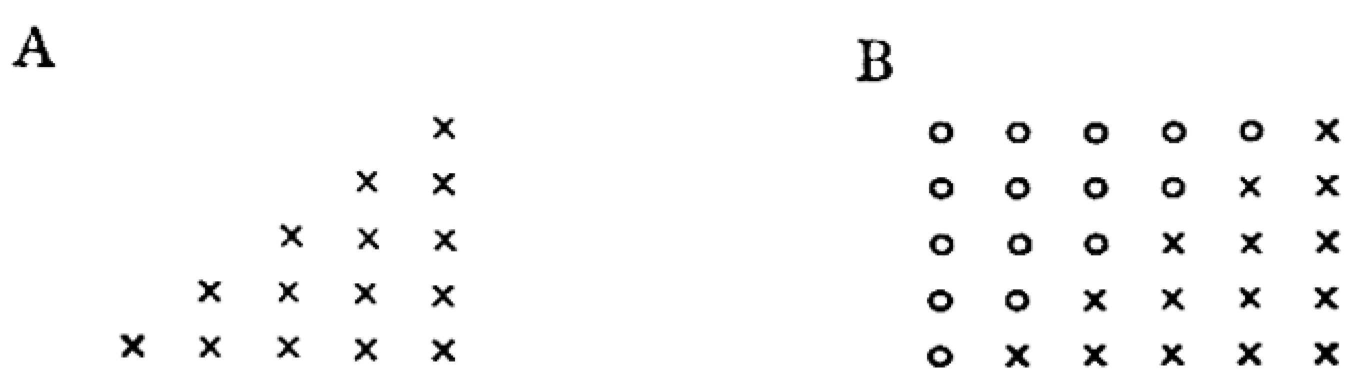

FIG. 38

Representing the first five terms as in the diagram A (Fig. 38) and comparing this with the diagram B, it will be seen that the noughts in B represent the same set of numbers as the crosses, so that B represents the series of 5 terms written once forward and once backward. Also it will be seen that twice the sum of the five terms is 5×6. This suggests that the general A.P. may be summed by writing it backwards as well as forwards and taking half the total.

S_n, the Sum to n terms

If l is written for the last term, the nth, $l = a + (n-1)d$ and the terms just before this are ... $l-3d$, $l-2d$, $l-d$. So writing the series both forwards and backwards:

$$S_n = a + (a+d) + (a+2d) + \dots \qquad + (l-2d) + (l-d) + l,$$
$$S_n = l + (l-d) + (l-2d) + \dots \qquad + (a+2d) + (a+d) + a.$$

If these are added, each term being taken with the one written above it,

$$2 \, . \, S_n = (a+l) + (a+l) + (a+l) + \dots \quad + (a+l) + (a+l) + (a+l).$$

Or, since there are n terms,

$$2 \, . \, S_n = n(a+l); \quad \therefore \; \mathbf{S_n = \tfrac{1}{2}n(a+l)}. \quad \dots\dots\dots\dots\text{(i)}$$

Writing $l = a + (n-1)d$, so that $a + l = 2a + (n-1)d$,

$$\mathbf{S_n = \tfrac{1}{2}n\{2a + (n-1)d\}}. \quad \dots\dots\dots\dots\dots\text{(ii)}$$

Either (i) or (ii) is used to find the sum of an A.P. to n terms.

Note that since $a + (a+d) + (a+2d) + \dots + \{a + (n-1)d\}$
$$= na + d\{1 + 2 + 3 + \dots + (n-1)\},$$
it will be sufficient to remember that $1 + 2 + 3 + \dots + (n-1) = \frac{1}{2}n(n-1)$ and the formula (ii) can be deduced.

Diagram to show S_n

The formula just found can be illustrated by the area diagram (Fig. 39).

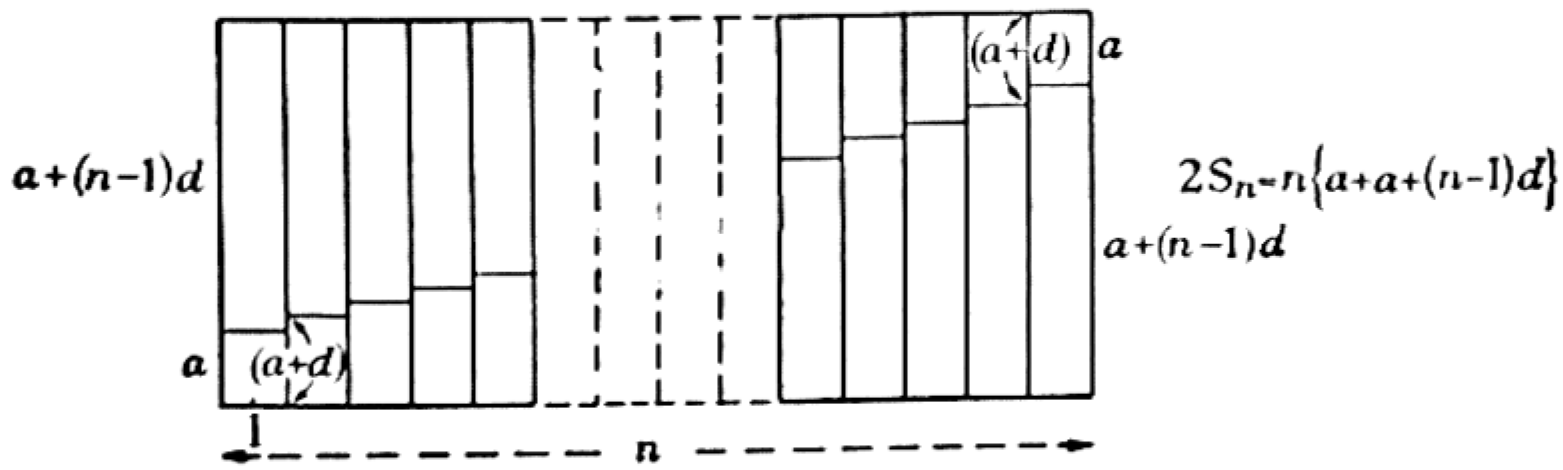

FIG. 39

Arithmetic Mean

If three numbers are in A.P., the middle one is called the *Arithmetic Mean* of the other two.

If a, x, b are in A.P., then

$$x - a = \text{common difference} = b - x;$$
$$\therefore\ 2x = a + b \quad \text{or} \quad x = \tfrac{1}{2}(a+b).$$

Thus the usual *average* of two numbers, as taught in Arithmetic, is the Arithmetic Mean of these numbers.

Example I. For the series $48 + 43 + 38 + 33 + \ldots$ find (i) the 12th term, (ii) the sum to 12 terms, (iii) the least number of terms which must be taken so that the sum is negative.

(i) Using the formula nth term $= a + (n-1)d$, since $a = 48$, $d = -5$, the 12th term $= 48 - 11 \times 5 = -7$.

(ii) Using the formula $S_n = \frac{1}{2}n(a+l)$ the sum to 12 terms $= 6\{48 + (-7)\} = 6 \times 41 = 246$.

(iii) In order that S_n may be negative, $a + l$ or $2a + (n-1)d$ must be negative; so $96 - 5(n-1)$ must be negative. Thus $n - 1$ must be at least 20 and $n = 21$ is the required number of terms.

Example II. In an Arithmetic Progression the fourth term is 7 and the 10th term 31. Find the first term and the common difference. Show that the sum to n terms is $2n^2 - 7n$, and find n so that this sum may be 2920.

If a is the first term and d the common difference, $a + 3d = 7$ and $a + 9d = 31$;

$$\therefore\ 6d = 31 - 7 = 24, \text{ and so } d = 4,\ a = -5,$$

$$S_n = \tfrac{1}{2}n\{2a + (n-1)d\} = \tfrac{1}{2}n\{-10 + 4n - 4\} = 2n^2 - 7n.$$

If $2n^2 - 7n = 2920$, $2n^2 - 7n - 2920 = 0$.

This gives $(n-40)(2n+73) = 0$; $\therefore\ n = 40$.

Example III. Prove that the sum of the squares of two numbers is always greater than twice the square of their arithmetic mean.

If the amount by which it is greater is 18, find the difference of the numbers.

Let the numbers and their arithmetic mean be

$$a - d, \quad a, \quad a + d.$$

It is usually simpler to take this form for three numbers in A.P. because it simplifies their sum. The question would have seemed harder if the numbers had been taken as a, b with $\frac{1}{2}(a+b)$ as their mean.

Then the sum of the squares of the numbers is

$$(a-d)^2+(a+d)^2=2a^2+2d^2$$

and twice the square of the arithmetic mean is $2a^2$.

Of these the former is greater by $2d^2$, which is always positive. If $2d^2=18$, then $d=\pm 3$ and the difference of the numbers is 6.

Note. The shift of accent in going from the noun to the adjective is interesting : geómetry, geométric ; aríthmetic, arithmétic ; hármony, harmónic.

Examples 59 : A.P.

1. (*a*) For the series $7+10+13+16+\ldots$ write down (i) the 10th term ; (ii) the nth term ; (iii) the sum to 10 terms ; (iv) the sum to n terms.
   (*b*) Repeat for the series $81+79+77+75+\ldots$.
2. Find the 15th term of the series $(-8)+(-4)+0+4+\ldots$ and the sum to 15 terms.
   If the sum to n terms of the series is zero, what is n?
3. Find the sum of the even numbers up to 100. By how much is the sum of the odd numbers up to 99 less than this?
4. The nth term of a sequence is $3n-22$. What is the least value of n for which this is one of the sequence of (positive) odd numbers?
5. (i) The first term of an A.P. is 31 and the common difference is 9 ; show that the nth term is $9n+22$, and find the sum to n terms.
   (ii) What are the first four terms of the series whose nth term is $5n+4$? Find the sum of the first 20 terms of this series.
6. Which is the number of the first term of the sequence 102, 105, 108, ... , that is greater than 200? In this sequence by how much is the 48th term greater than 200?
7. In an A.P. the 20th term is 200 and the common difference is -6. Find the 1st term and the sum to 20 terms.
8. In an A.P. the 9th term is 19 and the 15th term is 31. Find the 6th term.
9. The third term of an A.P. is -20 and the 11th term is $+20$. Show that the sum to thirteen terms is zero.
10. The nth term of the series $5+7+9+\ldots$ is the first that is greater than the nth term of the series $100+97+94+\ldots$. Find n.
11. The sum of $3n$ terms of the series $20+23+26+29+\ldots$ is equal to the sum of $2n$ terms of $a+(a+d)+(a+2d)+(a+3d)+\ldots$ for all values of n. Prove that $d=27/4$ and find a.

12. Find the arithmetic mean between

(i) 450 and -150; (ii) $(a+b)^2$ and $(a-b)^2$;
(iii) $(a+b)^2$ and $-(a-b)^2$; (iv) $(a+b)^3$ and $(a-b)^3$.

13. The A.M. of two numbers is $23\frac{1}{2}$ and their product is 510; find the numbers.

14. Show that the squares of x^2-2x-1, x^2+1, x^2+2x-1 are in A.P.

15. If a, b, c, d are four consecutive terms in A.P. show that $bc-ad$ must be positive. [Hint. Use $p-3q$, $p-q$, $p+q$, $p+3q$.]

16. Three cubes have the lengths of their edges in A.P. and the sum of these lengths is 12 inches. If the total volume of the cubes is 288 cubic inches find the lengths of the edges.

17. If the digging of an artesian well costs 2s. for the first foot, 2s. 6d. for the second, 3s. for the third, and so on, find the depth of the well if the whole cost is £2,035.

18. Draw the graph of $y=3+\frac{1}{2}x$ and indicate, by a set of small crosses on it, terms of an A.P. of values of y whose first term is $3\frac{1}{2}$ and common difference $\frac{1}{2}$. Find the sum to n terms of this A.P.

Geometric Progression (G.P.)

Here are some examples of the G.P.; each term in the series is formed from the previous one by *multiplying* it by the same number; that is, the *ratio* of any term to the previous one is the same number, called the *common ratio.*

$$3+6+12+24+48+\dots$$
$$1000+100+10+1+\tfrac{1}{10}+\dots$$
$$5-15+45-135+405-\dots$$

The common ratios in these cases are 2, $\frac{1}{10}$ and -3 respectively.

All these are included in the general case

$$a+ar+ar^2+ar^3+ar^4+\dots$$

where a is the first term and r the common ratio.

To get the nth term, notice the powers of r in the 4th and 5th terms; these are r^3 and r^4, the index being one less than the number of the term. Thus the nth term is ar^{n-1}.

The $n-1$ here should be compared with the $n-1$ in $a+(n-1)d$ for the A.P.

S_n, the Sum to n Terms

The device used to find a short expression for the sum is to

multiply S_n by r and write the terms on the R.H.S. under the like terms of S_n, thus

$$S_n = a + ar + ar^2 + ar^3 + \dots \qquad + ar^{n-2} + ar^{n-1}. \quad \dots\dots\dots\text{(i)}$$
$$rS_n = \qquad ar + ar^2 + ar^3 + \dots \qquad + ar^{n-1} + ar^n \dots\text{(ii)}$$

If $|r| < 1$, subtract (ii) from (i) to get

$$S_n - rS_n = a - ar^n \text{ or } S_n(1-r) = a(1-r^n).$$

$$\therefore\ S_n = \frac{a}{1-r} - \frac{ar^n}{1-r} \text{ or } S_n = \frac{a(1-r^n)}{1-r}. \quad \dots\dots\dots\text{(iii)}$$

If $|r| > 1$, subtract (i) from (ii) to get

$$S_n = \frac{ar^n}{r-1} - \frac{a}{r-1} \text{ or } S_n = \frac{a(r^n-1)}{r-1}. \quad \dots\dots\dots\text{(iv)}$$

As the case when $|r| < 1$ is perhaps the more important, it is usual to regard (iii) as the formula for the sum. The formula (iv) is the equivalent.

Alternatively, the known factors of $a^n - b^n$ may be used to find the sum.

Since $$1 - r^n = (1-r)(1 + r + r^2 + r^3 + \dots + r^{n-1}),$$

it is only necessary to multiply both sides by a to get the result (iii).

Geometric Mean

If three numbers are in G.P., the middle one is called the *Geometric Mean* of the other two.

If a, x, b are in G.P. and if r is the common ratio, then

$$x = ar \text{ and } b = ar^2;$$

$$\therefore\ x^2 = a^2r^2 = ab, \text{ so that } x = \pm\sqrt{(ab)}.$$ 16476

It is customary to regard $+\sqrt{(ab)}$ as *the geometric mean* of a and b.

In geometry this is usually called the *mean proportional.* It is here that it is easiest to see the connection of a geometric progression with geometry.

If a rectangle has sides a and b, the side of the square equal to it in area is $\sqrt{(ab)}$, the geometric mean of the sides.

Example I. For the geometric progression $4 + 3 + 9/4 + \dots$ find (i) the 6th term, (ii) the sum to 6 terms, (iii) the sum to 60 terms.

(i) For the series, $r = \frac{3}{4}$, so the 6th term is $4 \times \left(\frac{3}{4}\right)^5 = \frac{3^5}{4^4} = \frac{243}{256}$.

(ii) The sum to 6 terms $4\left\{\frac{1-(\frac{3}{4})^6}{1-\frac{3}{4}}\right\}=16\left\{1-\left(\frac{3}{4}\right)^6\right\}=16-\frac{3^6}{4^4}$

$$=16-\frac{729}{256}=16-2\frac{217}{256}=13\frac{39}{256}.$$

[The result for (i) and perhaps that for (ii) is found just as easily by writing down the 6 terms in order and, for (ii), adding them.]

(iii) The sum to 60 terms $=\frac{4}{1-\frac{3}{4}}-\frac{4}{1-\frac{3}{4}}\times\left(\frac{3}{4}\right)^{60}$

$$=16-16.\left(\frac{3}{4}\right)^{60}.$$

This is too troublesome to be worth working out to an accurate answer, so we use logarithms to obtain an approximate one :

$\log(\frac{3}{4})$	$=\bar{1}\cdot 8751$	$\bar{1}\cdot 8751$
$\log(\frac{3}{4})^{60}$	$=\bar{8}\cdot 506$	10
$\log 16$	$=1\cdot 204(1)$	$\bar{2}\cdot 751$
$\log 16\,.\,(\frac{3}{4})^{60}$	$=\bar{7}\cdot 710$	6
$16\,.\,(\frac{3}{4})^{60}$	$=5\cdot 13\times 10^{-7}$	$\bar{8}\cdot 506$

The required sum is best given as

$$16-5\cdot 13\times 10^{-7},$$

and $\simeq 16$ to at least 5 places of decimals.

Example II. The 5th term of a G.P. is 4 and the 10th term 128 ; find the common ratio and the first term and by how much the sum of the first seven terms is less than 32.

With the usual notation, $ar^4=4$, $ar^9=128$.

Dividing, $r^5=128/4=32$; $\therefore\ r=2$.

Since $a\,.\,2^4=4$, we get $a=\frac{1}{4}$.

The sum of the first 7 terms $=\frac{1}{4}+\frac{1}{2}+1+2+4+8+16=31\frac{3}{4}$.

This is short of 32 by $\frac{1}{4}$

$$\left[\text{or } S_7=\frac{1}{4}\frac{(2^7-1)}{2-1}=\frac{2^7}{4}-\frac{1}{4}=32-\frac{1}{4}\right].$$

Example III. The Geometric Mean of two numbers is 10 and the difference of the numbers is 21. If the numbers are positive, find them.

If a and b are the numbers, $\sqrt{ab}=10$ and $a-b=21$.

$\therefore\ ab=100$ and $a^2-2ab+b^2=441$, so that $a^2+2ab+b^2=841$.

$\therefore\ a+b=29$, which with $a-b=21$ gives $a=25$, $b=4$.

The numbers are 25 and 4.

Examples 60 : G.P.

1. For the following series, each of which is a G.P., state the common ratio and write down formula for

   (*a*) the 30th term; (*b*) the *n*th term; (*c*) the sum to 30 terms; (*d*) the sum to *n* terms.

   (i) 5, 10, 20, ... ; (ii) 72, 48, 32, ... ; (iii) 1, ·6, ·36, ...

2. What is the geometric mean between

   (i) 9 and 36 ; (ii) 1 and 1·21 ; (iii) a and ar^4 ; (iv) $(x+y)^2$ and $(x-y)^2$?

3. State the sum to n terms for the G.P.'s whose first two terms are

   (i) 12, 15 ; (ii) x^{-1} and x^2 ; (iii) ax and ax^4 ; (iv) x, $-\frac{1}{2}x^2$.

4. The third term of a G.P. is 24 and its seventh term is $4\frac{20}{27}$; find its second term.

5. Repeat No. 4 if the third term is $4\frac{20}{27}$ and the seventh term 24.

6. For the two G.P.'s $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$ and $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\dots$ obtain formulae for the sum to n terms and show that for even values of n the sum of the second series is one-third that of the first.

7. **Use the fact that $(a-b)^2>0$, unless $a=b$, to prove that**

$$\frac{1}{2}(a^2+b^2)>ab.$$

   Deduce that the A.M. of two positive numbers is greater than their G.M.

8. As in worked Example I (iii) find approximate answers to the questions :

   (i) By how much is the sum to 50 terms of the G.P. $5+2+4/5+\dots$ less than $8\frac{1}{3}$?

   (ii) By how much is the sum to 40 terms of the G.P. $3+{\cdot}3+{\cdot}03+\dots$ less than $3\frac{1}{3}$?

   (iii) By how much does the sum to 31 terms of the G.P.

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots \text{ differ from } \frac{2}{3}?$$

9. Show that the product of the first 10 terms of a G.P. of positive terms is equal to the product of the fifth and sixth terms raised to the fifth power.

10. If an A.P. has the same first two terms as a G.P. of positive terms, show that the third term of the G.P. is greater than the third term of the A.P.

11. If S_n is the sum of the first n terms of the series $a+ar+ar^2+\dots$ and S_n' is the sum of their reciprocals, show that $S_n=a^2r^{n-1}S_n'$.

12. If four terms are in G.P. show that the product of the first and last is equal to the product of the two middle ones. What similar statement can be made about eight terms in G.P.?

13. Find the common ratio of a G.P., three consecutive terms of which are the 1st, 2nd and 4th terms of an A.P.

14. In a G.P. of 7 terms, the sum of the last 6 terms is double the sum of the first 6 which is 378; what is the G.P.?

15. If a, b, na are in A.P., prove that a, b, $\frac{1}{2}(n+1)b$ are in G.P.

16. In any G.P., if the sum of the 1st $2n$ terms is p times that of the first n terms and the sum of $4n$ terms is q times that of $2n$ terms, prove that $(p-1)^2=q-1$.

17. ABC is a triangle right-angled at A; AD is the perpendicular from A to BC. In this figure point out geometric means of (i) BD, DC; (ii) BD, BC; (iii) CD, CB.

Harmonic Progression. (H.P.)

The reciprocals of a series of numbers in arithmetic progression are said to be in *harmonic* progression. Here are some examples of H.P.s:

$$\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\dots;\quad \tfrac{12}{2}+\tfrac{12}{5}+\tfrac{12}{8}+\tfrac{12}{11}+\dots,\ \text{i.e. } 6,\ 2\tfrac{2}{5},\ 1\tfrac{1}{2},\ \tfrac{12}{11},\ \dots,$$

and the general case $\dfrac{1}{a}, \dfrac{1}{a+d}, \dfrac{1}{a+2d}, \dots$.

For the nth term we take the nth term of the A.P. and then take its reciprocal; for the third series above, the nth term is $1/\{a+(n-1)d\}$.

There is no formula for the sum to n terms of an H.P.

Harmonic Mean. (H.M.)

Most of the applications of H.P. are concerned with H.P.s of *three terms only* and with the relation of two terms to their harmonic mean.

Suppose a, c, b to be in H.P.; then $\dfrac{1}{a}, \dfrac{1}{c}, \dfrac{1}{b}$ are in A.P.;

$$\therefore\ \frac{2}{c}=\frac{1}{a}+\frac{1}{b}\ \text{(I)} \quad \text{which gives} \quad c=\frac{2ab}{a+b}\ \text{(II)}.$$

These formulae (I), (II) are the important ones to remember about H.P.

Those who have studied the reflection of light by a spherical mirror will have met a formula connecting the distances u and v of the object and image from the mirror with its radius r.

The formula is $\frac{1}{u}+\frac{1}{v}=\frac{2}{r}$, so u, r, v are in H.P.

Instances of three lengths in H.P. are extraordinarily frequent in simple geometrical figures.

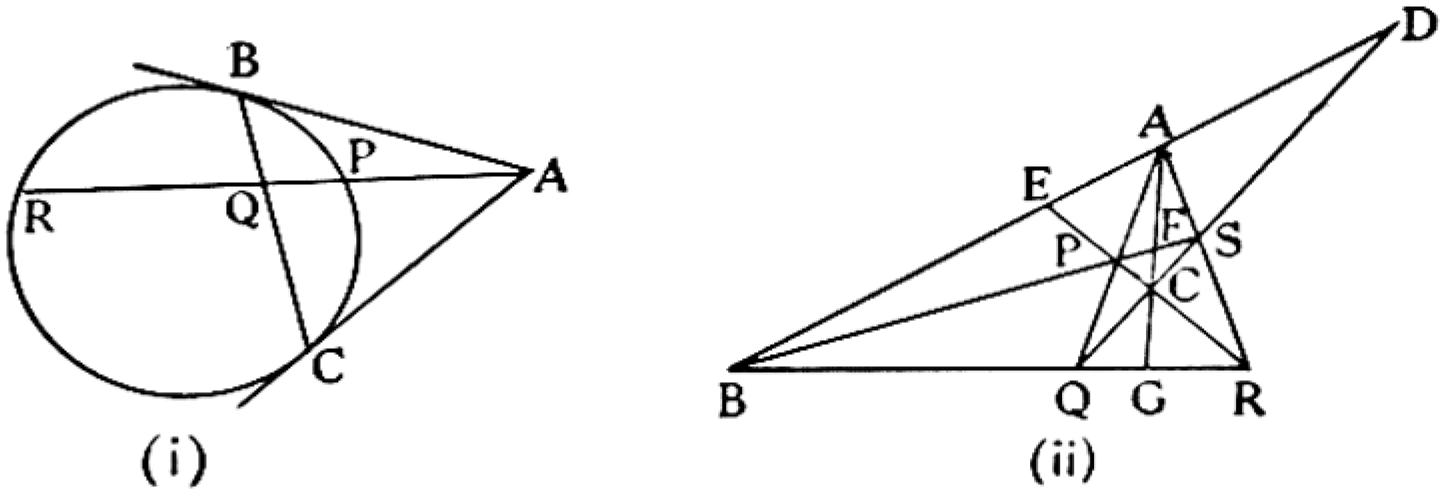

FIG. 40

For instance, in Fig. 40 (i) where AB, AC are tangents to a circle, *any* line through A cutting the circle and BC is cut so that AP, AQ, AR are in H.P., or as it is more usually expressed, AQ is the H.M. between AP, AR, or $APQR$ is divided harmonically.

Again, in Fig. 40 (ii) when P, Q, R, S are *any* four points joined up as shown, *every* line in the figure is divided harmonically, e.g. BG is the H.M. between BQ, BR ; AC is the H.M. between AF, AG ; RP the H.M. between RC, RE, and so on.

The name harmonic progression is derived from the fact that if the strings in a stringed instrument such as a harp (or a piano) give an even scale, their lengths are in harmonic progression.

Examples on H.P. are usually worked by considering first the corresponding A.P., or if there are only 3 terms concerned, by using the formulas (I) or (II).

Example I. The first two terms of an H.P. are 2 and $\frac{4}{3}$; find the 5th term and the nth term.

The corresponding A.P. has $\frac{1}{2}$ and $\frac{3}{4}$ as its first two terms and so $\frac{1}{4}$ as common difference. Of this A.P. the 5th term is $\frac{1}{2}+4\cdot\frac{1}{4}=\frac{3}{2}$ and the nth term is $\frac{1}{2}+(n-1)\frac{1}{4}=\frac{n+1}{4}$.

$\therefore$ for the H.P. the 5th term is $\frac{2}{3}$ and the nth term $\frac{4}{n+1}$.

Example II. In an H.P. the first term is -1 and the 6th term is $\frac{3}{7}$; show that the fourth term is 1 and find the number of the first term which is less than $\frac{1}{16}$.

For the corresponding A.P. the first term is -1 and the 6th term $\frac{7}{3}$.

$\therefore$ If d is the common difference, $5d=\frac{7}{3}-(-1)=\frac{10}{3}$; $\therefore$ $d=\frac{2}{3}$.

The A.P. is $-1, -\frac{1}{3}, \frac{1}{3}, \frac{3}{3}, \frac{5}{3}, \ldots$.

$\therefore$ The H.P. is $-1, -3, 3, 1, \frac{3}{5}, \ldots$, and the fourth term is 1.

A term of the H.P. $<\frac{1}{15}$, if the corresponding term of the A.P. >15.

The nth term of the A.P. is $-1+(n-1)\frac{2}{3}=\dfrac{2n-5}{3}$.

$$\frac{2n-5}{3}>15 \quad \text{if} \quad 2n-5>45 \quad \text{or} \quad 2n>50 \quad \text{or} \quad n>25.$$

[The 25th term is $-1+\dfrac{24\times 2}{3}=15$; the 26th and subsequent terms will be greater than this.]

The first term of the H.P. which is less than $\frac{1}{15}$ is the 26th term.

Example III. If a, b, c are in G.P., show that the ratio of the H.M. of b and c to the H.M. of a and b is $b : a$.

If r is the common ratio of the G.P. we have $b=ar$ and $c=ar^2$.

The H.M. of b and c is $\dfrac{2bc}{b+c}=\dfrac{2a^2r^3}{ar(1+r)}=\dfrac{2a^2r^2}{a(1+r)}$.

The H.M. of a and b is $\dfrac{2ab}{a+b}=\dfrac{2a^2r}{a(1+r)}$.

The ratio of the first H.M. to the second is seen to be r, which is $b : a$.

Examples 61

1. The first two terms of an H.P. are $1\frac{1}{2}$ and $\frac{3}{4}$; show that the sixth term is $\frac{1}{4}$ and find the twelfth term.
2. The third and fourth terms of an H.P. are 12 and 3; find the first term and show that the sixteenth term is $\cdot 3$.
3. A, B are two points on a line 3 in. apart. Points P and Q on the line divides AB internally and externally in the ratio $2:1$. Show that AP, AB, AQ are in harmonic progression, and so also are QB, QP, QA.
4. Generalise No. 3 with AB x inches and P, Q dividing AB internally and externally in the ratio $\lambda : \mu$.
5. AOB is a diameter of a circle centre O; X and Y are taken on AB so that $OX \,.\, OY=OB^2$. Show that AB is the harmonic mean of AX and AY.
6. If $a>b>c$, and a, b, c are in harmonic progression, show that $\dfrac{a}{c}=\dfrac{a-b}{b-c}$. Show that this is also true if $a<b<c$.

   [This property has sometimes been taken as the definition of harmonic progression.]

7. Find the arithmetic mean, the geometric mean and the harmonic mean of the pairs of numbers (i) 4, 9; (ii) $p-q$, $p+q$, where p, q are positive and $p>q$.

Verify that the A.M. > the G.M. > the H.M. in each case.

8. If $b+c$, $c+a$, $a+b$ are in A.P., prove that a, b, c are in A.P., but if $b+c$, $c+a$, $a+b$ are in H.P., prove that a^2, b^2, c^2 are in A.P.

9. If A, G, H are the arithmetic, geometric and harmonic means between a and b, express each of them in terms of a and b and prove that

$$G^2=AH \text{ and } A>G>H.$$

10. If a_1, a_2, a_3 ... are in H.P., prove that

$$a_1a_2+a_2a_3+a_3a_4+\ldots+a_{n-1}a_n=(n-1)a_1a_n.$$

[Hint: Add first two, then add third, and so on.]

The Sum of a G.P. to a large number of Terms

The sum to 50 terms of the G.P. $2+\frac{4}{3}+\frac{8}{9}+\ldots$ has a short *symbolic* answer given by the formula $\dfrac{2\{1-(\frac{2}{3})^{50}\}}{1-\frac{2}{3}}$ or $6-6(\frac{2}{3})^{50}$. This appears quite short, but to work it out accurately is scarcely practicable, and in any case would be a complete waste of time. The best we can do is to find an *approximate* value for $6(\frac{2}{3})^{50}$ using logarithms. This is $9{\cdot}55\times10^{-9}$, as seen by the work at the side, and the required sum

$$\simeq 6-9{\cdot}55\times10^{-9}.$$

$$50\times\log(\tfrac{2}{3})=50\times\bar{1}{\cdot}8240,$$
$$\simeq -50+41{\cdot}20\simeq\bar{9}{\cdot}20,$$
$$\log\{6\times(\tfrac{2}{3})^{50}\}\simeq{\cdot}78+\bar{9}{\cdot}20$$
$$=\bar{9}{\cdot}98.$$

When the sum of a G.P. is asked for to such a number of terms as 40 or 50 terms it must be assumed that an approximate answer, to be obtained by using logarithms, is wanted.

Such sums are of two classes according as r is numerically greater than 1 or less than 1.

If $r>1$, $-\dfrac{ar^n}{1-r}$ or $\dfrac{ar^n}{r-1}$ is so big that $\dfrac{a}{1-r}$ hardly counts; while if $r<1$ numerically, $-\dfrac{ar^n}{1-r}$ is so small that it hardly counts and the sum is very nearly $\dfrac{a}{1-r}$.

Thus in the worked example above, it is shown that the sum of the series $2+\frac{4}{3}+\frac{8}{9}+\ldots$ to 50 terms differed from 6, which was the value of $\dfrac{a}{1-r}$ by a term approximately equal to $9{\cdot}55\times10^{-9}$.

For this reason, and because by going to a still larger number of terms we can make the difference as small as we please, and feel sure that if we go even further the difference will become even smaller, the number 6 is called the *limiting sum* of the series.

It is also called "*the limit of the sum*" or "*the sum to infinity*" and we can write $S_n \to 6$ as $n \to \infty$, where "$\to$" is read "tends to".

Positive and negative values of r

In the series just mentioned, the sum to any number of terms is always *less* than the limiting sum, but this is not the case if r is negative.

Consider, for instance, the G.P. $4-3+\frac{9}{4}-\dots$, in which $a=4$, $r=-\frac{3}{4}$, so that $\dfrac{a}{1-r}=\dfrac{4}{1+\frac{3}{4}}=\dfrac{16}{7}$.

Here $S_n=\frac{16}{7}-\frac{16}{7}(-\frac{3}{4})^n$.

The limiting sum is $16/7$. If $n=60$ the sum is less than $16/7$, but if $n=61$ the sum is greater than $16/7$, the difference in each case being very small.

The Modulus notation

It is convenient to use the modulus notation $|r|$ for the *numerical value* of r.

What has been said above can be summarised by saying:

If $|r|>1$ the sum of a G.P. to n terms becomes very large as n becomes large; but if $|r|<1$ the sum of a G.P. to n terms, as n becomes large, approaches very closely the value $\dfrac{a}{1-r}$, which is called the limiting sum or sum to infinity. The series is said to *converge*.

If $|r|=1$ there are the two cases $r=1$ and $r=-1$.

In the first case the series is $a+a+a+a+\dots$, of which the sum can be made as large as is wished by increasing the number of terms. The series is said to *diverge*.

In the second case the series is $a-a+a-a+a-a+\dots$, of which the sum to any *odd* number of terms is a, while the sum to any *even* number of terms is zero. The series is said to *oscillate*.

How $|r^n|$ diminishes if $|r|<1$

If $|r|<1$, multiplication by r must always diminish a number numerically, but it might be supposed wrongly that if $|r|$ is nearly 1 the value of $|r^n|$ would not ever become really small; but in fact it always does so.

If, for example, $r = \cdot 9$ we can show that r^n becomes quite small while n is still less than 150.

To show that $(\frac{9}{10})^n$ is very small, is the same as to show that $(\frac{10}{9})^n$ is very large.

Now $\log \frac{10}{9} = \log 10 - \log 9 = 1 - \cdot 9542 \simeq \cdot 0458$ which $> \cdot 04$.

So $\log (\frac{10}{9})^n > n \times \cdot 04$.

If we wish to make $(\cdot 9)^n$ less than $\frac{1}{10^5}$ or $\cdot 00001$ we must make $(\frac{10}{9})^n$ greater than 10^5 and therefore $\log (\frac{10}{9})^n > 5$.

Now $n \times \cdot 04 > 5$ if $n > \frac{5}{\cdot 04}$, i.e. $> 5 \times 25$ or 125.

So $(\frac{9}{10})^n$ will be less than $\frac{1}{10^5}$ if $n = 126$ or a greater number.

(It need not really be so great, for the $\cdot 0058$ has been left out, but in such questions *it is not necessary to find the least possible value of n* unless this is specially asked for.)

That $x^n \to 0$ as $n \to \infty$ if $0 < x < 1$ may be pròved thus:

$$nx^n < x + x^2 + \ldots + x^n = \frac{x}{1-x} - \frac{x^n}{1-x} < \frac{x}{1-x}.$$

$$\therefore x^n < \frac{1}{n} \cdot \frac{x}{1-x} \text{ which } \to 0 \text{ as } n \to \infty.$$

For extension to the case of nx^n see Ex. 68, No. 41.

[Proof due to R. L. Goodstein, 1954.]

Recurring Decimals

Recurring decimals are instances of G.P.'s which have limiting sums; for instance, $\cdot\dot{7} = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$ and the sum to infinity is $\frac{7}{10} \div (1 - \frac{1}{10}) = \frac{7}{9}$.

$\cdot\dot{2}\dot{3} = \frac{23}{10^2} + \frac{23}{10^4} + \frac{23}{10^6} + \ldots$ and the sum to infinity is

$$\frac{23}{100} \div (1 - \frac{1}{100}) = \frac{23}{99}.$$

Formulae for G.P.

For the G.P. $\mathbf{a + ar + ar^2 + ar^3 + \ldots}$

$$\mathbf{S}_n = \frac{\mathbf{a}}{1-\mathbf{r}} - \frac{\mathbf{ar}^n}{1-\mathbf{r}} \quad \text{or} \quad \mathbf{S}_n = \frac{\mathbf{ar}^n}{\mathbf{r}-1} - \frac{\mathbf{a}}{\mathbf{r}-1}.$$

If $|\mathbf{r}| < 1$ the series has a limiting sum or sum to infinity.

The sum to infinity $= \frac{\mathbf{a}}{1-\mathbf{r}}$.

The difference between S_n and the sum to infinity is $\dfrac{ar^n}{1-r}$ and if $|r|<1$, $r^n \to 0$ as $n \to \infty$.

It is sufficient to remember the sum for $1+r+r^2+\ldots+r^{n-1}$, which is $(1-r^n)\div(1-r)$. For the general G.P. it is only necessary to multiply by a.

Example I. If the sum to infinity of $1+r+r^2+r^3+\ldots$ (r being positive and less than 1) is double the sum to infinity of $1-r+r^2-r^3+\ldots$, find the value of r and show that, in this case, if n is even S_n for the first series is double S_n for the second. What happens if n is odd?

It is given that $\dfrac{1}{1-r}=2\cdot\dfrac{1}{1+r}$; $\therefore$ $1+r=2-2r$; $\therefore$ $r=\frac{1}{3}$.

S_n for the first series is $\frac{3}{2}\{1-(\frac{1}{3})^n\}$ and for the second $\frac{3}{4}\{1-(-\frac{1}{3})^n\}$.

If n is even the first of these is double the second, but if n is odd

$$1-(-\tfrac{1}{3})^n>1,$$

and the first S_n is less than double the second.

Example II. Find the value of recurring decimal $\cdot 219\dot{4}\dot{5}$.

The first three figures give $\dfrac{219}{10^3}$, the next two $\dfrac{45}{10^5}$, and the rest

$$\frac{45}{10^7}+\frac{45}{10^9}+\ldots.$$

For G.P., starting with the $\dfrac{45}{10^5}$, the sum is given by $\dfrac{a}{1-r}$ where

$$a=\frac{45}{10^5} \quad\text{and}\quad r=\frac{1}{10^2}.$$

$$\therefore \text{ the sum is } \frac{45}{10^5}\div\left(1-\frac{1}{10^2}\right)=\frac{45}{10^3}\times\frac{1}{99}.$$

$$\text{So the value is } \frac{219}{1000}+\frac{45}{99000}=\frac{21945-219}{99000}=\frac{21726}{99000}.$$

[If a rule has been learnt in Arithmetic for dealing with such decimals, this is the point at which to see that the rule agrees with the above result.

Another proof is this:

If $x=\cdot 219454545\ldots$,

then $10^5\cdot x=21945\cdot 454545\ldots$ } both decimals going on

and $10^3\cdot x=\quad 219\cdot 454545\ldots$ } indefinitely.

Subtracting: $(10^5-10^3)x=21945-219$

$=21726$;

$\therefore$ $99000x=21726$.

The same result as before.]

Example III. The first two terms of an infinite G.P. are together equal to 1 and every term is twice the sum of all the terms that follow; find the series.

With the usual notation the first condition is

$$a + ar = 1.$$

The second condition is $a = 2\dfrac{ar}{1-r}$, $ar = 2\dfrac{ar^2}{1-r}$, ..., all of which follow from the first, which is $a - ar = 2ar$.

This gives $r = \frac{1}{3}$, and then the other gives $a = \frac{3}{4}$.

So the series is $\frac{3}{4} + \frac{1}{4} + \frac{1}{12} + \dots$.

Examples 62 [The series are G.P.'s.]

1. If $a = 6$ and $|r| = \cdot 4$, show that the limiting sum of the G.P. is 10 or $4\frac{2}{7}$ according as r is positive or negative.

Find approximately by how much the sum to 50 terms differs from the limiting sum in each case.

2. How large must n be so that (i) $(\cdot 7)^n < \cdot 0005$, (ii) $(\frac{1}{4})^n < \cdot 0001$?

3. Find approximately the 100th term of the sequence 12, 18, 27, ..., and the 50th term of the sequence 36, 24, 16, ...

4. Show that the 15th term of the series $16 + 8 + 4 + 2 + \dots$ is less than $\cdot 001$. Write down a formula for the sum to 15 terms of this series, and determine (i) the sum to infinity, (ii) the amount by which the sum to 15 terms falls short of the sum to infinity.

5. Find approximate values for the sum to 80 terms of the series $8 + 24 + 72 + \dots$ and to 150 terms of $108 + 36 + 12 + \dots$.

6. Show that the 150th term of the series $1 + 1\cdot 2 + 1\cdot 44 + \dots$ is greater than 5×10^{11}, while the sum of these 150 terms is less than 5×10^{12}.

7. Express each of the following recurring decimals as fractions:

(i) $\cdot 1\dot{4}$; (ii) $\cdot\dot{2}0\dot{3}$; (iii) $\cdot 4\dot{0}\dot{7}$.

8. How many terms of the series $1 + \cdot 6 + (\cdot 6)^2 + (\cdot 6)^3 + \dots$ must be taken so that the sum may be greater than $2\cdot 49$? Can you by taking further terms make the sum greater than $2\cdot 5$?

9. Find the sum to $2n$ terms and the sum to infinity of

$$4 - 2 + 1 - \tfrac{1}{2} + \tfrac{1}{4} - \dots .$$

10. If $3\sqrt{2}$ and $2 - \sqrt{2}$ are the first two terms of a G.P., show that the third term is $\frac{1}{3}(3\sqrt{2} - 4)$ and that the sum to infinity is $\dfrac{9(2\sqrt{2}+1)}{7}$.

11. The inventor of the game of chess is said to have claimed as his reward a single grain of corn for the first square of the chessboard, two for the second, four for the third, and so on in G.P. up to the 64th square.

Find approximately the total number of grains of corn claimed.

12. Find a formula for the sum of the series $a + ax + ax^2 + ax^3 + \dots$ starting at the 101st term as far as the 150th term.

If $x < 1$, what is the limiting sum of the terms of the series after the 150th term?

Compound Interest and G.P.

If a sum of money is allowed to accumulate at compound interest, the values it reaches at the end of 1, 2, 3, 4, ... years form a sequence of terms in G.P.

Thus £1 accumulating at 3% compound interest reaches at the end of 1, 2, 3, 4, ... years, the values £1·03, £$1{\cdot}03^2$, £$(1{\cdot}03)^3$, £$(1{\cdot}03)^4$, ... and £P reaches values P times these.

If £P has to be paid *now* in order that £100 may be received 10 years hence we must have £$P \times (1{\cdot}03)^{10} =$ £100 or $P = 100 \times (1{\cdot}03)^{-10}$. £$P$ is called the *present value*.

These ideas are of importance in reckoning what lump sum must be paid now to ensure an annuity later on.

Example : How much must be paid in order to receive £100 a year for 7 years, the first payment to be received 10 years hence, if compound interest is reckoned at $2\frac{1}{2}$%?

For the 1st £100 to be paid 10 years hence the present value is £$100 \times (1{\cdot}025)^{-10}$, for the next £$100 \times (1{\cdot}025)^{-11}$, and so on.

The total value is

$$£100 \times \left\{1{\cdot}025^{-10} + 1{\cdot}025^{-11} + \dots\ 1{\cdot}025^{-16}\right\}$$ which written in reverse order

$$= \frac{£100}{1{\cdot}025^{16}}\{1 + 1{\cdot}025 + \dots + 1{\cdot}025^6\}$$

$$= \frac{£100}{1{\cdot}025^{16}}\left\{\frac{(1{\cdot}025)^7 - 1}{1{\cdot}025 - 1}\right\}$$

$$\simeq £4{,}000\left\{\frac{1{\cdot}189 - 1}{1{\cdot}025^{16}}\right\}$$

$$\simeq £756 / 1{\cdot}025^{16}$$

$$\simeq £509.$$

1·025	·0107239
$1{\cdot}025^7$	·0750673
756	2·8785
$1{\cdot}025^{16}$	·1715824
509·2	2·7069

Thus £510 now should secure the 7 payments of £100.

Note. Annuities usually are contingent on the age and end with the death of the annuitant : so the expectation of life at various ages is taken into account by Insurance Companies when fixing the purchase price of annuities.

Examples 63

1. A father, wishing his son to receive a payment of over £500 on his 21st birthday, pays on his behalf £25 annually from his first to his 20th birthday. If interest is added at $2\frac{1}{2}\%$ compound, how much should the son receive? If the payment was to be exactly £500, at the same rate of interest, how much should the father pay each year?
2. Savings Certificates bought at 15s. each are to be worth 20s. 6d. in ten years' time. What rate of compound interest is this?
3. A firm credits each employee with £50 deferred pay at the end of each completed year of service. If these sums earn interest at 4% p.a find the amount of deferred pay expected by a man who worked 35 years for the firm. [$\log 1{\cdot}04 = {\cdot}0170333$.]
4. The amounts at compound interest of £2,400 at 5% after 1, 2, 3 ... years form a G.P. Write down the nth term of this G.P. The logarithms of these amounts form an A.P. Give the first term and common difference of this A.P. and write down its nth term.
5. £2,500 is invested at 4% compound interest and is to be repaid in 15 equal annual payments, the first instalment to be paid in 15 years' time. Find the amount of this annual repayment.

Exponential Graph and G.P.

Ordinates of an exponential graph such as $y = 1{\cdot}2^x$ (Fig. 41) at points evenly spaced along the x-axis form a sequence of terms in G.P.

A simple case is given by

$$\begin{array}{lllllll} x = & 0 & 1 & 2 & 3 & 4 & \dots \\ y = & 1 & 1{\cdot}2 & 1{\cdot}2^2 & 1{\cdot}2^3 & 1{\cdot}2^4 & \dots \end{array}$$

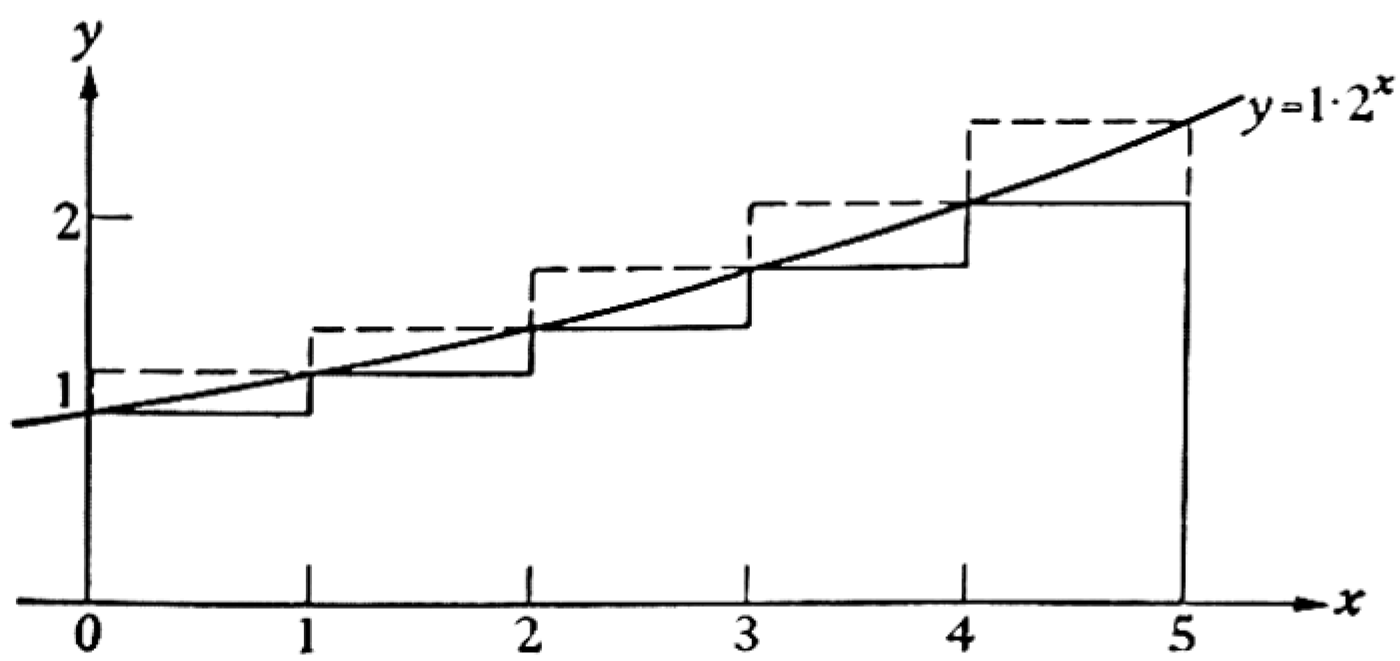

FIG. 41

Sum of G.P shown as an area

The sum $1 + 1{\cdot}2 + 1{\cdot}2^2 + 1{\cdot}2^3 + 1{\cdot}2^4$ is shown above as the area enclosed by the black line.

If instead we take the area up to the dotted line it is

$$1\cdot 2 + 1\cdot 2^2 + 1\cdot 2^3 + 1\cdot 2^4 + 1\cdot 2^5.$$

The area below the curve, denoted by $\int_0^5 1\cdot 2^x \, dx$, is intermediate between these two.

This illustrates the way in which sums of G.P.'s may be used to obtain approximations to the area below an exponential curve.

Examples 64

1. In the above, calculate the two sums mentioned and thus obtain two values between which the integral lies.
2. If in the above ordinates were drawn when $x = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots$, find the two sums corresponding to those in No. 1 and so get two numbers closer to the value of the integral.
3. By using a graph in the same way as in Fig. 41, show that $\int_0^4 1\cdot 1^x \, dx$ lies between 4·64 and 5·11.
4. By using the graph of $y = 1/x$ in the same way as in Fig. 41, show that $\int_1^4 \frac{1}{x} \, dx$ lies between $1 + \frac{1}{2} + \frac{1}{3}$ and $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.
5. Sketch the graph of $y = 1\cdot 2^{-x}$ and by the method used above show that $\int_0^4 1\cdot 2^{-x} \, dx$ lies between 2·58 and 3·11.

Averages

It is usual to call the Arithmetic Mean of two *numbers* their *average*; when the average of two *quantities* is required, however, it will not always be given by finding the Arithmetic Mean.

Consider the following examples:

Example I. Two things cost, one a shilling and the other eightpence; what is the average price?

If each had cost 10d., the total cost would have been the same.

Thus the *average* price is $\frac{1}{2}(12 + 8)$ pence, the *arithmetic mean* of the two prices.

Example II. A cyclist rides 24 miles with the wind at 12 mi./hr. and returns against it at 8 mi./hr.; what is his average speed?

Here the *average* speed is *not* $\frac{1}{2}(12 + 8)$ mi./hr.

The whole trip of 48 miles takes 2 hours out and 3 hours back, in all 5 hours.

If the speed had been $\frac{48}{5}$ mi./hr. all the way, the total time would have been the same.

Thus the *average* speed is $\frac{2 \times 12 \times 8}{12+8}$ mi./hr., the *harmonic mean* of the two speeds, for this is $\frac{48}{5}$.

Example III. The population of a town increases each five-year period by the same percentage of its value at the beginning of the period.

In 1920 the population was 100,000, and in 1930 was 121,000. What was its population in 1925, and what is the average rate % of increase for a five-year interval?

Here we should find the population in 1925, P_{25}, thus:

$$P_{30} : P_{25} = P_{25} : P_{20} ;$$

hence P_{25} is the *geometric mean* of P_{30} and P_{20} and is

$$\sqrt{[100{,}000 \times 121{,}000]} = 110{,}000.$$

In 5 years the increase is from 100 to 110, an increase of 10%.

Here the *geometric mean* is used in finding the *average* rate.

Examples 65

1. For a journey of x miles at u mi./hr. followed by one of x miles at v mi./hr., show that the average speed is the H.M. of u and v.
2. For a journey of y hours at u mi./hr. followed by one of y hours at v mi./hr., show that the average speed is the A.M. of u and v.
3. If $5xy$ things are bought at x for a shilling and another $5xy$ at y for a shilling, what is the average number bought for a shilling?
4. If p shillings are spent on things at x for a shilling and another p shillings on things at y for a shilling, what is the average number bought for a shilling?

   The first lot being bought as before, if only q shillings are spent on the things at y for a shilling, what is the average number bought for a shilling?

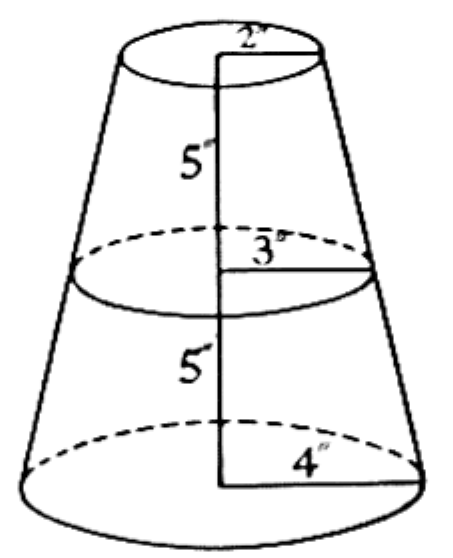

FIG. 42

5. If equal sums of money are spent on apples at u for 1s. and on apples at v for a shilling, show that the average number of apples bought per shilling is the A.M. of u and v, but that if equal numbers of apples are bought at these two prices the average number of apples per shilling is the H.M. of u and v.
6. Find the average cross-section of a frustum of a cone (Fig. 42) whose base is a circle of radius 4 inches, whose top is a circle of radius 2 inches, and whose height is 10 inches.

   [The average cross-section is that cross-section which if constant would give the same volume.]

Verify that the result can be found by applying *Simpson's Rule*, which states that

Average section $= \frac{1}{6}\{$sum of end-sections $+ 4 \times$ mid-section$\}$.

Here the mid-section is 9π, and the rule gives

$$\frac{1}{6}\left\{16\pi + 4\pi + 4 \times 9\pi\right\} = \frac{56\pi}{6}.$$

7. If 3 boys are of height 4 ft. 10 in. and 5 others of height 5 ft. 2 in., by how much does the average height exceed 5 ft.?
   [It is not necessary to find the total height of all the boys.]
8. A fruiterer buys 50 lb. of apples at 8d. a lb. and 40 lb. of apples at x pence a lb. If the average cost of all the apples is 7d. a pound, find x.
9. A fruiterer buys 50 lb. of apples at 8d. a lb. and y lb. of apples at 5d. a lb. If the average cost of all the apples is 7d. a pound, find y.

Means for more than Two Numbers

When more than two numbers are concerned, the word "mean" is understood in two different senses, but there is not much likelihood of confusion between them.

If $a_1, a_2, a_3, \ldots, a_n$ are in A.P., then the terms $a_2, a_3, \ldots, a_{n-1}$ are said to be $n-2$ Arithmetic Means between a_1 and a_n.

If a_1 and a_n are given, the others are easily found, for d, the common difference, is given by $a_n = a_1 + (n-1)d$, and then the other terms are found from $a_2 = a_1 + d$, $a_3 = a_1 + 2d$, and so on. (See Examples 66.)

The other, and much more important, sense of the word "mean" is for the A.M. of any n numbers $a_1, a_2, \ldots, a_n$, namely

$$\frac{1}{n}(a_1 + a_2 + a_3 + \ldots + a_n).$$

Thus the A.M. of n numbers is what is usually called their average.

The same usage of "means" as intermediate terms of a series is customary for the G.P. and the A.P.

The Geometric Mean of any n numbers $a_1, a_2, \ldots, a_n$ is defined to be

$$+\sqrt[n]{(a_1 a_2 a_3 \ldots a_n)},$$

i.e. the positive nth root of their product.

The Harmonic Mean H of $a_1, a_2, \ldots, a_n$ is given by the equation

$$\frac{n}{H} = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}.$$

Examples 66

1. Insert 6 arithmetic means between 2 and 23.
2. Insert 5 geometric means between 2 and 128.
3. Insert 3 harmonic means between $\frac{1}{3}$ and $\frac{1}{11}$.
4. Insert 5 arithmetic means between a and b and show that the third of them is $\frac{1}{2}(a+b)$.
5. Insert two geometric means between a^3 and b^3.
6. Insert three harmonic means between 3 and 5.
7. Prove that the arithmetic mean of n arithmetic means between a and b is the arithmetic mean between a and b.
8. If $a_1, a_2, \ldots a_9$ are in A.P., show that the A.M. of a_1 and a_9 is
$$\tfrac{1}{7}(a_2+a_3+\ldots+a_8).$$
9. If $a_1, a_2, a_3 \ldots a_9$ are in G.P. and a_2, a_5, a_8 are in H.P., show that the terms of the G.P. are all equal. [Common ratio 1.]

Calculation of the Mean

The arithmetic mean of a set of numbers is the mean most frequently employed ; unless stated to the contrary, the A.M. is intended when the single word " mean " is used.

In the statistical examination of a set of numbers, the calculation of the mean is generally the first step.

It should be realised that sets of figures differ in character ; for instance, the number of cars passing a certain point on a road between 12 noon and 1 p.m. is a definite number, but the height of a man who passed along the road could only be given to a certain degree of accuracy ; a height which lay between 5 ft. $9\frac{3}{4}$ in. and 5 ft. $10\frac{1}{4}$ in. would probably be recorded as 5 ft. 10 in. If the heights of a group of people were being recorded it is probable that several would be given the same height, although it is unlikely that any two of them had exactly the same height.

The number of times a measurement or the result of an observation is repeated is called the *frequency*.

The following table gives the heights of the men in a certain infantry platoon.

Height - -	5′ 7″	5′ $7\frac{1}{2}$″	5′ 8″	5′ $8\frac{1}{2}$″	5′ 9″	5′ $9\frac{1}{2}$″	5′ 10″
Number of men (Frequency) -	2	5	6	9	7	2	2

The straightforward method of finding the mean is to obtain the total height of all the men (i.e. the sum of the products of each height by its frequency) and then divide by 33 (the number of men).

Alternatively, the mean of the number of inches by which each height exceeded 5 ft. could be found. (This is equivalent to taking a new zero)

i.e. $\{7 \times 2 + 7\frac{1}{2} \times 5 + 8 \times 6 + 8\frac{1}{2} \times 9 + 9 \times 7 + 9\frac{1}{2} \times 2 + 10 \times 2) \div 33$
$= 278/33 = 8{\cdot}424$;

$\therefore$ the average height is 5 ft. 8·424 in.

Another arrangement, and the one most frequently used in statistical work, is given below ; the *class-number* 0 (zero) is assigned to one height (or *class*), and class numbers $+1$, $+2$ ..., -1, -2 ... given to the other classes.

The work of the previous example is arranged as follows :

Height	*Frequency* (f)	*Class-number* (x)	$f \cdot x$
5′ 7″	2	-3	-6
5′ 7½″	5	-2	-10
5′ 8″	6	-1	-6
5′ 8½″	9	0	
5′ 9″	7	$+1$	$+7$
5′ 9½″	2	$+2$	$+4$
5′ 10″	2	$+3$	$+6$
	33		-22 $+17$ $= -5$

x of Mean (denoted by $\bar{x}$) $= \dfrac{-5}{33} = -{\cdot}152$.

$\therefore$ the average height is ·152 class-intervals less than 5 ft. 8½ in. (in this example the class-interval is ½ in.)

$\therefore$ the average height $= 5$ ft. $8\frac{1}{2}'' - {\cdot}076''$
$= 5$ ft. $8{\cdot}424''$.

This last method does not show up to advantage when the numbers involved are few as here, but it will be found to be far less laborious than other methods in cases where there are many recordings. (See No. 5 below.)

Middle of Class

In the example just given, the measurements are supposed to be taken to the nearest half-inch. The 5′ 8″ or 68″ class includes the men whose height lies between $67\frac{3}{4}''$ and $68\frac{1}{4}''$, and of this class the middle point is 68″ exactly.

It is however more usual to name a class from its starting point. Thus the classes might be

height 68″ and not as much as 69″, denoted by 68″–
height 69″ and not as much as 70″, denoted by 69″–

and the table above would be shortened to

Class	67″–	68″–	69″–	70″–
f	7	15	9	2

Here measurements having been taken to the nearest half-inch, the class 68– really extends, not from 68″ but from $67\frac{3}{4}''$ to $68\frac{3}{4}''$, and its middle point is not $68\frac{1}{2}''$, as might be supposed, but $68\frac{1}{4}''$.

If, however, measurements had been taken to the nearest $\frac{1}{4}$ inch, the 68″– class would extend from $67\frac{7}{8}''$ to $68\frac{7}{8}''$, with mid-point $68\frac{3}{8}''$.

It will be noticed that with this greater accuracy of measurement, a man whose height lay between $67\frac{3}{4}''$ and $67\frac{7}{8}''$ would be transferred from the 68″– class to the 67″– class.

Statistics

The work of these last two pages may be regarded as a first lesson in the subject of *Statistics*. Those who require further work in this subject—work dealing with dispersion, correlation, sampling, etc.—are referred to books dealing solely with this subject.

Examples 67

1. Find the average height of the 33 men in the example worked above by assigning the class-number zero to height (i) 5 ft. 8 in., (ii) 5 ft. $9\frac{1}{2}$ in.
2. Eggs are bought at the prices shown: find the average price per dozen.

Price per dozen -	2/6	2/9	3/0	3/3	3/6
No. of dozen -	20	25	28	19	12

3. The ages of boys in a school were recorded in class-intervals of 6 months, e.g. ages from 15 years 3 months to just less than 15 years

9 months being recorded as $15\frac{1}{2}$ years. The following table was produced.

Age	11	$11\frac{1}{2}$	12	$12\frac{1}{2}$	13	$13\frac{1}{2}$
Number of boys	32	40	45	47	50	48

Age	14	$14\frac{1}{2}$	15	$15\frac{1}{2}$	16	$16\frac{1}{2}$
Number of boys	43	46	38	41	37	23

Find the average age by assigning class-numbers taking (i) the $13\frac{1}{2}$ class as zero, (ii) the 14 class as zero.

4. In a recent examination the marks awarded to the first 20 scripts were 32, 57, 43, 65, 28, 60, 47, 52, 39, 48, 25, 53, 47, 52, 62, 31, 38, 46, 72, 51. The marks can be put into classes 25–29, 30–34, 35–39, etc., these classes being indicated by 25 +, 30 +, 35 +, etc., to give the following table:

Marks	25 +	30 +	35 +	40 +	45 +
No. of scripts	2	2	2	1	4

Marks	50 +	55 +	60 +	65 +	70 +
No. of scripts	4	1	2	1	1

Find the mean by the ordinary method and also by assigning class-numbers (note that the class 40 + (i.e. 40–44) has 42 as its centre, and there are five marks between the centres of adjacent classes).

5. The heights of a number of men were classified to give the following table, in which 60– means a height between 60 and 61 inches.

Height, inches	60–	61–	62–	63–	64–	65–
Frequency	41	83	169	394	669	990

Height, inches	66–	67–	68–	69–	70–	71–
Frequency	1223	1329	1230	1063	646	392

Find the mean height, taking the middle of the 67– class, for instance, as 67·5 inches.

6. Show that the middle of a class depends on the accuracy with which measurements are being taken by showing that in No. 5, if measurements are to the nearest $\frac{1}{4}$ inch, the middle of the 67– class is $67\frac{3}{8}$, but, if measurements are to the nearest $\frac{1}{8}$ inch, it is $65\frac{7}{16}$.

[Hint. Measuring to nearest $\frac{1}{4}$ inch, all heights between $64\frac{7}{8}$ and 65 are counted in the 65– class.]

Examples 68. Miscellaneous Examples on the Progressions

1. If four numbers are in A.P. prove that the product of the two middle ones is greater than the product of the two end ones.

If the first, second and fourth of these numbers are in G.P. prove that the common ratio of the G.P. must be 2. (B.)

2. Show that if a, b, c are in arithmetic progression and a, b, t are in geometric progression, then

$$4at = (a+c)^2. \qquad \text{(B.)}$$

3. Prove that of the squares of three consecutive numbers the middle one is the A.M. between the A.M. and G.M. of the two end ones.

Prove that the same is true if the three numbers squared are in A.P.

4. In an arithmetical progression whose first term is a and common difference d, write down four consecutive terms, the first of the four being the nth term of the series.

Show that the product of the second and third of these terms is greater than the product of the first and fourth by $2d^2$. (L.)

5. If s is the sum to n terms of an A.P. of first term a and common difference d, show that

$$n^2d - n(d - 2a) - 2s = 0.$$

Hence, if a and d are integers, prove that $(d - 2a)^2 + 8sd$ is a perfect square.

If $s = 247$, $a = 1$ and $d = 3$, verify that this is so and find n. (L.)

6. If the $(n-1)$th and $(n+1)$th terms of a G.P. are the A.M. and the H.M. between a and b, prove that the nth term is $\sqrt{ab}$.

7. The A.M. between a and b is $(1+x^2)^2$ and the H.M. is $(1-x^2)^2$. Find the G.M. and determine a and b.

8. If two numbers a and b are taken to be the areas of two circles whose radii are 17 in. and 19 in., verify that the arithmetic mean of $\log a$ and $\log b$ is the logarithm of the geometric mean of a and b.

Verify, with the above values of a and b, that their geometric mean is less than their arithmetic mean and that the area of a circle of radius 18 in. is intermediate in value between these two means (B.)

9. If the mth and nth terms of an A.P. are in the ratio $2m-1 : 2n-1$, prove that the sum of the first m terms is the sum of the first n terms as m^2 is to n^2. (O. & C.)

10. (i) Find the sum of n terms of the series whose nth term is $\frac{23}{10^{2n}}$.

(ii) Express $0\cdot\dot{2}\dot{3}$ as a vulgar fraction.

(iii) Find the sum of n terms of the series

$$\cdot 23 + \cdot 2323 + \cdot 232323 + \cdot 23232323 + \ldots,$$

by first finding $(1 - 10^{-2})$ times this sum. (B.)

11. An Arithmetic Progression and a Geometric Progression each have p as first term and q as second term, where $q<p$. Write down the expression for s, the sum to infinity of the G.P., and prove that the sum of n terms of the A.P. may be written as

$$np - \frac{n(n-1)}{2}\frac{p^2}{s}. \qquad \text{(L.)}$$

12. Find the ratio of two numbers for which the A.M. : the G.M. as $5:4$.

13. Find an A.P. such that S_n is $(n+1)$ times $\frac{1}{2}u_n$.

14. Prove that $bc-a^2$, $ca-b^2$, $ab-c^2$ are in A.P. if a, b, c are in A.P.

15. There are p arithmetic progressions, each beginning with unity, and the common differences are $1, 2, 3, \ldots p$. Show that the sum of their nth terms is $\frac{1}{2}\{(n-1)p^2+(n+1)p\}$.

16. Assuming that the value of a machine depreciates each year by an amount which is $r\%$ of its value at the beginning of the year, obtain an expression for its value at the end of n years if its present value is £A.

Taking $A=3000$, $r=12$, calculate (i) the value when $n=10$ (ii) in how many years the value will be reduced to one quarter of its present value. (L.)

17. Find the sum of n terms of an Arithmetical Progression whose first term is a, second term b.

Find the sum of all the integers between 1 and 200, excluding those that are multiples of 3 or 7.

18. Sum the series $1-2+3-4+\ldots$ to $2n$ terms

(i) by taking it as $1+2+3+4+\ldots-\{2+4+6+\ldots\}$,

(ii) by taking the terms in pairs.

19. A journey consists of a miles travelled at u miles an hour, followed by b miles at v miles an hour and c miles at w miles an hour. Find the average speed.

If the average speed is v miles an hour and $u=\frac{1}{2}v=\frac{1}{4}w$, show that $c=2a$.

20. If A, G, H are the arithmetic, geometric and harmonic means of a and b and if $A = kH$, prove that $A^2 = kG^2$.

Find the ratio of a to b if $k = \frac{3}{2}$.

21. A set of 100 drives with a golf ball were classified as follows:

Length in yards	150–	160–	170–	180–	190–	200–	210–	220–	230–
Frequency	4	16	9	21	23	14	7	5	1

Find the average length of these drives, taking the mid-point of class 150– as 155, etc.

22. Marks in an Algebra paper obtained by candidates from one school taking a certificate examination were distributed as follows:

Percentage Mark	10–	20–	30–	40–	50–	60–	70–
Frequency	2	4	8	7	21	6	8

Find the average mark.

23. Find the sum of n terms of the series

$$2 + \frac{4}{3} + \frac{8}{9} + \dots .$$

Show that however great n be, the sum can never exceed 6; find the least number of terms whose sum exceeds 5·99. (L.)

24. Find the sum of the first six terms of the geometric series whose third term is 27 and whose sixth term is 8.

Find how many terms of this series must be taken if their sum is to be within $\frac{1}{10}\%$ of the sum to infinity. (L.)

25. The sum of n terms of a series is $3n^2$ for $n = 1, 2, 3, \dots$. Find the rth term of the series. What is the first term?

Calculate the sum of all numbers between 0 and 201 which are multiples of 5 or 7; that is, find the sum of the series

$$5 + 7 + 10 + 14 + 15 + \dots + 35 + \dots + 200. \qquad \text{(O. \& C.)}$$

26. Find the sum of nine terms of a geometrical progression of which the 4th term is 7 and the 7th term is 4.

The amplitudes of the oscillations of a pendulum diminish in geometrical progression, the ratio of consecutive amplitudes being 0·975. If the amplitude of the 50th swing is 1°, find after what swing the amplitudes are less than $2'$. (L.)

27. Find the sum of n terms of a Geometrical Progression whose first two terms are a and b.

A pendulum swings in such a way that the amplitude of each swing is 0·89 of the amplitude of the preceding swing. If the amplitude of the first swing is 15° find after how many swings the amplitude will be less than 1°. Find the sum of the amplitudes of the first 9 swings. (L.)

28. A man borrows £1,000 and agrees to repay it by 10 *equal* annual payments, the first payment being made at the end of one year and interest being charged at the rate of 6% per annum, calculated annually, on that part of the sum which is not yet repaid. Find the value of the annual payments. (L.)

29. A manufacturer orders machine plant to the value of £5,000. It is agreed that £2,000 shall be paid on delivery and that the balance of the debt shall be discharged by three *equal* annual instalments, the first instalment becoming due at the end of one year from the date of delivery. Find the amount of each instalment, interest being reckoned at 5% per annum. (L.)

30. The nth term of a series is $2^{n-1}+2n$. Find the sum of the first n terms and evaluate your expression when $n=10$. (L.)

31. Find the sum of n terms of the series

$$a+ax+ax^2+ax^3+\dots.$$

The cost of replacing a machine is estimated at £3,500. What annual sum paid into a fund and taking compound interest at 4% per annum will suffice to replace the machine after 15 years? [The first repayment is to be made at the end of the first year.] (L.)

32. Find the sum of n terms of the series

$$a+ar+ar^2+ar^3+\dots.$$

Prove that the sum of n terms of the series

$$5+55+555+5555+\cdots$$

is $\frac{5}{81}[10^{n+1}-10-9n]$. (L.)

33. If S_n denotes the sum of the geometric series

$$1-\tfrac{2}{3}+\tfrac{4}{9}-\dots,$$

show as a graph with n as abscissa and S_n as ordinate the sum of n terms for values of n from 1 to 6. Take 1 inch as unit for n and 5 inches as unit for S_n.

Find the least number of terms of the series whose sum differs from the sum to infinity by less than 10^{-4}. (L.)

34. Find the sum of the first n terms of the series

$$1+x+x^2+x^3+\dots.$$

A man has initially £P invested in a security bearing r% interest per annum. At the end of each year he draws the interest and sells sufficient stock to make the total sum withdrawn up to £p. Prove that after n years the amount of his capital remaining is

$$PR^n-p\frac{R^n-1}{R-1}$$

where $R=1+r/100$, and show that if $p=P/20$ and $r=4$ he can continue for 41 years without his capital becoming exhausted. (L.)

35. A solid is formed by fastening together ten solid circular cylinders, each of height one inch, whose radii are in arithmetical progression. The radius of the smallest is 3 in. and that of the largest is 15 in. The cylinders are placed with their axes coincident and each one rests on the next larger one. Find the total surface area of the solid. (L.)

36. The number of new motor-cars produced by a certain manufacturing firm is the same in each year, whereas every year 15% of all the cars existing at the end of the previous year are scrapped. At the beginning of a certain year there are 10,000 of the firm's cars on the road; at the end of the tenth year after this there are 15,000. Find the number of new cars produced by the firm each year. (L.)

37. Find the present value of an annuity of £A to continue for n years, interest being reckoned at r% per annum.

A man owns the lease of a house which has 45 years to run; the rent he receives for the house is £150 per annum, the repairs and other expenses amount to £25 per annum, and the ground rent is £20 per annum. Assuming interest at 6%, find the present value of his lease. (L.)

38. Obtain the formula for the sum of n terms of a geometric progression with a as first term and r as common ratio.

If $S = a + (a+d)r + (a+2d)r^2 + \dots + (a + \overline{n-1}d)r^{n-1}$, prove that

$$(1-r)S = a + dr\frac{1-r^{n-1}}{1-r} - (a + \overline{n-1}d)r^n.$$

Hence find the sum of the first 10 terms of the series

$$1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots. \qquad \text{(L.)}$$

39. Prove that the purchase price of an annuity of £A payable for n years, the first instalment being due m years hence, is

$$\frac{A(R^n - 1)}{R^{m+n-1}(R-1)},$$

where R is the amount of £1 at the end of one year. (L.)

40. If the birth and death rates in a country remain constant at 20 and 15 per thousand respectively, and if there is no emigration or immigration, prove that the populations at annual intervals form a geometric progression, and find the percentage increase in population in 100 years.

If all the people live to the same age x, i.e. if all who are born in any one year die x years later, show that

$$1{\cdot}005^x = 4/3,$$

and thence find x. (L.)

41. If $0 < x < 1$ it has been proved (p. 159) that

$$nx^n < \frac{x}{1-x}. \tag{i}$$

Taking x' between x and 1, replace x in (i) by $\frac{x}{x'}$, and show that $nx^n < \frac{x}{x'-x} \cdot x'^n$.

Deduce that $nx^n \to 0$ as $n \to \infty$.

[Due to R. L. Goodstein.]

CHAPTER VIII

PERMUTATIONS; COMBINATIONS; BINOMIAL THEOREM; PROBABILITY

Arrangement of Permutations

For a given number of objects, in how many ways can they be arranged in order?

To answer this question it is well to take a few special cases and then attempt general reasoning.

Represent the objects by letters and consider in how many ways 3 or 4 letters can be arranged in order.

The three letters H, A, T of the word HAT may be arranged in the six ways HAT, HTA, ATH, AHT, THA, TAH. It will be noticed that with any one of the letters in the first position there are two ways of arranging the other two letters, and so there are 3×2 *arrangements* altogether.

For the four letters of the word HATE, the number of arrangements of the letters is obtained by recognising that with any one of the *four* letters in the first place there are 3×2 ways of arranging the remaining *three* letters.

$\therefore$ there are $4 \times 3 \times 2$ ways of arranging the *four* letters.

Again, the *five* letters of the word HATED could be rearranged in $5 \times 4 \times 3 \times 2$ ways, since with any one of the *five* letters in the first position the other *four* letters could be arranged in $4 \times 3 \times 2$ ways.

A permutation is an arrangement of *some* or *all* of a number of things; so the number of *permutations* of *all* the letters of the word HATED is $5 \times 4 \times 3 \times 2$ or $5 \times 4 \times 3 \times 2 \times 1$.

The shorthand way of writing this product $5 \times 4 \times 3 \times 2 \times 1$ is $5!$ or $\underline{|5}$, and either of these is read as *factorial* 5.

Again, $\underline{|7}$ or $7!$ stands for $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

It is usual to think of $5!$ as the product of *five* factors,

$$5 \times 4 \times 3 \times 2 \times 1.$$

With this definition it is seen that $1! = 1$.

The argument above can be extended to show that the number of permutations of n different things is $n!$.

Examples 69

1. Find the number of permutations of all the letters in each of the following words:

(i) PLACE; (ii) GANDER; (iii) HALTING; (iv) DOWNRIGHT.

2. In how many ways may six men and six ladies form couples for a dance?

3. Simplify (i) $\dfrac{7! \times 4!}{10!}$; (ii) $\dfrac{\underline{|12}}{\underline{|10}}$; (iii) $\dfrac{n!}{(n-2)!}$; (iv) $\dfrac{(n+1)!}{(n-1)!}$.

4. In how many ways can a batting order be drawn up for a cricket eleven?

5. How many different 5-figure numbers can be made from the figures 8, 6, 4, 2, 0 (zero must not appear in the first place)?

6. Find the values of:

(i) $9! \div 8!$; (ii) $n! \div (n-3)!$;
(iii) $n! \div n(n-1)(n-2)$; (iv) $(10!\, 8!\, 4!) \div (7!\, 6!\, 5!)$;
(v) $10! \div (5!)^3$; (vi) $(3!)^5 \div (5!)^3$.

Case when some Letters are repeated

As an example, contrast the possible arrangements of the letters of the word SPEED with those of the word SPEND, for which the number is $5!$.

Any one arrangement of the letters of SPEED, for example SEPED, will, without altering the positions of S, P and D give two arrangements, SEPND and SNPED, of the letters of SPEND.

It follows that the number of ways of arranging the letters of SPEED is exactly half the number for those of SPEND, that is $5! \div 2$, or which is the same thing, $5! \div 2!$.

Again, compare the arrangements of the letters of BRASSES with those of the letters of BRACKET.

Choosing any one set of positions for B, R, A and E, say EB . R . . A using the letters of BRASSES, we must fill the three vacant places with S in each place and get one arrangement, EBSRSSA.

But since the three letters C, K, T can be arranged in $3!$ or six ways, with E, B, R, and A in the same positions we get the 6 possible arrangements:

EBCRKTA, EBCRTKA, EBTRCKA, EBTRKCA, EBKRCTA, EBKRTCA.

Thus each arrangement of the letters of BRASSES gives 6 or $3!$ arrangements for those of BRACKET, for which the arrangements total $7!$.

$\therefore$ $7!$ is six times the number of arrangements for the letters of BRASSES. These can therefore be arranged in $7! \div 3!$ ways.

Similar reasoning will show that the letters in PAINTING can be permuted in $8! \div (2 \times 2)$ ways, while the letters in PALATABLE can be permuted in $9! \div (2!\,3!)$ ways.

As the things concerned need not be letters, it is usual to speak, not of *repeated* letters, but of *like* things, or of *n things of which p are alike*. The general case is:

Find the permutations of n things, when p are alike of one kind, q alike of another kind, etc.

Let x be the required number. With any one permutation, if the p like things were replaced by p unlike things, these unlike things could be permuted among themselves in $p!$ ways to give $p!$ different permutations without changing the positions of the other things. So this change would result in the total number of permutations being $x \,.\, p!$.

In the same way if the q like things were then changed to q unlike things these could be permuted in $q!$ ways and the new total would be $x \,.\, p!\,q!$.

If this process were carried on till all the groups of like things had been changed to unlike things, all the n things would be unlike and the number of permutations would be $n!$.

$$\therefore\ x \,.\, p!\,q!\,r! \ \ldots = n!$$

$$\therefore\ x = \frac{n!}{p!\,q!\,r!\ \ldots}\,.$$

Examples 70

1. Find the number of permutations of the letters, taken all together, in

   (i) SCHOOL; (ii) HARASS; (iii) ISOSCELES;
   (iv) MISSISSIPPI.

2. Seven peas and five beans are arranged in a row. If no distinction is made between any two of the peas or between any two of the beans, how many arrangements are possible?

3. A bag contains 2 white balls, 5 red balls, 3 blue balls and a green ball. Find the number of different orders of the colours when the balls are taken from the bag one at a time and not replaced.

Forecasting football matches

The first match (A versus B) may have 3 results: (i) a win for A, (ii) a draw, (iii) a win for B.

Similarly the second match may have 3 results, any one of which may occur with any one of the 3 results of the first match.

$\therefore$ there are $3 \times 3 = 3^2$ ways of forecasting the first two matches.

Again, the third match may have 3 results, any one of which may occur with any one of the 9 possible results of the first two matches.

$\therefore$ there are $3 \times 3 \times 3 = 3^3$ ways of forecasting the first three matches.

This argument can be extended to show that there are 3^{10} ways of forecasting 10 matches.

Similarly, when a penny is tossed, the result may be a head or a tail (H or T). If it is tossed twice the possible results are HH, HT, TH, TT, the number of which is 2^2; if thrice, the number is 2^3, and so on.

Examples 71

1. A boy writes out 100 ways in which the results of 5 football matches may occur. Show that he has not written half the possible results.
2. A man with 10 friends decides not to dine alone, but to have one or more of his friends to join him. How many different dinner parties are possible? [He may, or may not, ask each one.]

Nos. 3, 4 are preliminary for the next page.

3. Seven steamers travel between England and Ireland. In how many ways is it possible to cross by one steamer and return by a different one?
4. There are 5 routes by which a motorist can go from Winchester to Bournemouth. In how many ways can he go by one route and return by a different one?
5. How many different signals of 5 symbols can be sent using the dot and dash of the Morse code? How many if 5 symbols or less may be used?
6. There are three post pillars in a road. In how many ways may 12 letters be posted in the road?
7. A die with 6 sides numbered one to six is thrown 10 times and the results of the separate throws written down. Show that there are more than 6×10^7 possible results.

Permutations when all the things are not used

Given a number of objects, it may be required to find the arrangements of a smaller number selected from them.

For example, given 7 letters, a, b, c, d, e, f, g, it may be asked in how many ways it is possible to fill 3 places, with 3 of the letters.

The first place can be filled in 7 ways.

When this has been filled in any one way, the second place can be filled in 6 ways. Hence the first two places can be filled in 7×6 ways.

When two letters have been selected to fill the first two places in any one of these 7×6 *ways*, the third place can be filled in 5 ways. Hence the three places can be filled in $7 \times 6 \times 5 = 210$ ways.

This number is usually called the number of "*permutations*" of 7 things 3 at a time and is denoted by the symbol ${}_7P_3$.

Note that in ${}_7P_3$ the last factor is $7-2$ or $7-3+1$.

What has been shown on p. 177 is that ${}_4P_4 = 4!$, ${}_5P_5 = 5!$, ${}_nP_n = n!$.

General Case

To find the number of permutations of n different things taken r at a time, ${}_nP_r$.

The first position may be filled in n ways.

Having filled the first position, $(n-1)$ things are left, and so the second position can be filled in $(n-1)$ ways; i.e. with each of the n ways of filling the first position the second position can be filled in $(n-1)$ ways.

$\therefore$ there are $n(n-1)$ different ways of filling the first *two* places. When the first two positions are filled, $(n-2)$ things are left from which to choose the thing to occupy the third position.

$\therefore$ there are $n(n-1)(n-2)$ different ways of filling the first *three* places.

Similarly there are $n(n-1)(n-2)(n-3)$ ways of filling the first four places.

Notice that the last bracket $(n-3) = (n - \textit{four} + 1)$.

Extending the above argument, the number of different ways of filling the first r places is $n(n-1)(n-2)(n-3)\ldots(n-r+1)$.

$$\therefore\ {}_nP_r = \frac{n!}{(n-r)!}.$$

Note that when all the things are used, the number of permutations is $n!$ To make the formula just given fit this case also, we must *define* $0!$ to mean 1.

Examples 72

1. Write down the different ways of filling the first two places with two different letters chosen from the five letters a, b, c, d, e. Show that the number agrees with the formula $5! \div 3!$.

2. Write down the formula for ${}_nP_5$ in factors and also using factorials. What is the last factor in the formula for ${}_nP_7$?

3. Which is the greater, ${}_{81}P_2$ or ${}_{20}P_3$, and by how much?

4. Write down in factorial form the values of ${}_{10}P_3$, ${}_5P_4$, ${}_{20}P_7$, ${}_{18}P_9$.

5. Show that ${}_nP_r \cdot {}_{(n-r)}P_s = {}_nP_s \cdot {}_{(n-s)}P_r$ and that each is equal to ${}_nP_{(r+s)}$.

   [Of course, n must be not less than $r+s$.]

6. Find n if ${}_{2n}P_2 - 1 = 5({}_nP_2 - 1)$.

7. In how many ways can a forward line of 5 players be arranged if there are 8 forwards to choose from? How is this number reduced if 2 of the players can only play outside right and 1 can only play centre?

8. On a bookshelf there are 26 books, 6 bound in red, 5 in blue, 5 in black and the rest in green. In how many ways can the books be arranged on the shelf if books of any one colour are kept together?

9. Find the number of ways in which 30 books may be arranged on a shelf if two particular books are kept apart.

10. A, B, C, D enter a railway carriage in which there are eight seats; in how many ways can they sit (i) if there is no restriction? (ii) if A must sit facing the engine? (iii) A and D do not sit directly facing each other?

11. The "registration number" of a car is SLJ953. How many different "registration numbers" would be made from these letters and numbers, if the letters must come first?

    If the above were replaced by SL4953, how many possibilities would there be?

12. The ten figures 1, 2, 3, ... 9, 0 are each used once to form a ten-figure number. Show that $40 \times 8!$ of these numbers are odd.

Selections or Combinations

The question of the number of ways in which a number of objects may be *selected* from a larger number is closely connected with the previous work.

Suppose, for example, that it is required to find in how many ways 3 of the 7 letters a, b, c, d, e, f, g can be chosen, no regard being had to their order.

If we compare the required number with the ${}_7P_3$ permutations of the 7 letters 3 at a time, we see that for any *selection* of 3 letters, say a, c, e, there are 6 or $3!$ *arrangements* to be included in these ${}_7P_3$ permutations.

It follows that the number of selections is ${}_7P_3 \div 3!$

This number is usually called the number of *combinations* of 7 things taken 3 at a time and is denoted by the symbol ${}_7C_3$ or ${}^7C_3$.

We therefore have the result

$${}_7C_3 = {}_7P_3 \div 3\,! = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}.$$

The argument given above is clearly general.

For each selection or combination included in ${}_nC_r$ there will be $r!$ permutations included in ${}_nP_r$.

$$\therefore\ {}_nC_r = {}_nP_r \div r\,! = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots r} = \frac{n\,!}{r\,!\,(n-r)\,!}.$$

Another way of getting the formula for ${}_nC_r$ is suggested in Ex. 9.

Notice that ${}_7C_3 = {}_7C_4$, since if any 3 of the 7 letters , a, b, c, d, e, f, g, say a, c, e have been chosen, there are 4, viz. b, d, f, g left, and we might have chosen the 4 to be left out in place of the 3 to be taken.

Similar reasoning shows that ${}_nC_r = {}_nC_{n-r}$. This also is clear from the formula for ${}_nC_r$.

Examples 73

1. Write down (i) the combinations, (ii) the permutations of the letters a, b, c, d taken 3 at a time, arranging them so as to show that 6 permutations correspond to each combination. Of what general result is this a particular case?
2. Do the same as in Example 1 for the five letters a, b, c, d, e taken two at a time. Also write down the combinations of these five letters three at a time, and arrange so as to show why ${}^5C_2 = {}^5C_3$.
3. Calculate ${}^{20}C_2$, ${}^{10}C_3$, ${}^{11}C_9$, ${}^{14}C_{11}$.
4. Out of 14 players, in how many ways can a cricket eleven be chosen
   (i) if no places in the team have been filled ;
   (ii) if 8 places have been already filled?
5. Find the number of ways of choosing three letters from the names
   (i) SOUTHEND ; (ii) CROYDON ; (iii) LONDON.
6. Write out the values of

   $$\text{(i) } {}^7C_r \text{ for } r = 1, 2, 3, \ldots 6,$$
   $$\text{(ii) } {}^8C_r \text{ for } r = 1, 2, 3, \ldots 7,$$

   and hence explain in general terms how ${}^nC_r$ changes for values $r = 1, 2, \ldots n-1$, pointing out when there is one value of r for which ${}^nC_r$ is greatest, and when there are two such values.
7. In how many ways can 3 factors be selected from the product $(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n)$?
   Evaluate this number if $n = 15$.

8. Write down the combinations of a, b, c, d, e taken 3 at a time, and show that the number of those that do not contain a is ${}^4C_3$, while the number of those that contain a is ${}^4C_2$.

By generalising this result prove that

$$ {}^nC_r = {}^{\overline{n-1}}C_r + {}^{\overline{n-1}}C_{r-1}. $$

Verify this by using the formulae for ${}^nC_r$, etc.

9. Write down the combinations 2 at a time of a, b, c, d: from each one (such as ab) form the 3-at-a-time combinations got by adding one of the other letters (such as abc, abd), and show that the 3-at-a-time combinations will by this process be obtained 3 times over.

Show that ${}^nC_r = \dfrac{n-r+1}{r} \cdot {}^nC_{r-1}$. How can the formula for ${}^nC_r$ be obtained from this result?

10. In the number of ways in which 6 articles can be chosen from 10 different articles what is the number of combinations in which a particular thing occurs?

11. In ${}^9C_3$ show that the number of combinations in which a particular thing occurs is one-third of the whole number.

Show that the same is true for ${}^{3n}C_n$.

12. Find n (i) if ${}_nC_3 = 6 \cdot {}_{(n-1)}C_2$; (ii) if ${}_nC_4 = 5 \cdot {}_{(n-2)}C_3$.

13. In how many ways can a committee which is to consist of 3 masters and 3 boys be chosen from 15 masters and 30 boys?

14. How many of the committees which could be chosen in No. 13 would not contain a particular master and a particular boy together? (i.e. the master may be on the committee but not the boy, and vice versa). Give a symbolic answer.

15. A newspaper holds a Christmas competition in which 10 suitable songs are to be selected from a list of 24 songs. In how many ways is it possible to make the selection?

Alternative Notation

In ${}_nC_r$, the number of combinations of n things taken r at a time, both n and r are necessarily positive integers; but in the *formula* for ${}_nC_r$, though r *must* be a positive integer, n *need not* be so. The expression is much used, e.g. in the general Binomial Theorem, without the restriction on the value of n; so an alternative notation is introduced.

We define $\quad \dbinom{n}{r} = \dfrac{n(n-1)(n-2)\dots(n-r+1)}{1 \cdot 2 \cdot 3 \dots r}.$

The denominator is usually written $r!$.

Note that, since in general $r! = r\,.\,(r-1)!$, it is convenient to define $0!$ as equal to 1, so that $1! = 1\,.\,0!$. (Also see p. 181.)

In the same way we define $\binom{n}{0} \equiv 1$, which fits with $0! = 1$.

It has been shown that $\binom{n}{r} = \binom{n}{n-r}$ if n is a positive integer, $n > r$, but if n is not a positive integer greater than r, no meaning can be assigned to the R.H.S., $(n-r)$ being negative or fractional.

Example. Show that $\binom{8}{4} + \binom{8}{3} = \binom{9}{4}$.

Explain by considering selections.

Solution.
$$\binom{8}{4} + \binom{8}{3} = \frac{8\,.\,7\,.\,6\,.\,5}{1\,.\,2\,.\,3\,.\,4} + \frac{8\,.\,7\,.\,6}{1\,.\,2\,.\,3}$$
$$= \frac{8\,.\,7\,.\,6\,.\,5 + 8\,.\,7\,.\,6\,.\,4}{1\,.\,2\,.\,3\,.\,4}$$
$$= \frac{9\,.\,8\,.\,7\,.\,6}{1\,.\,2\,.\,3\,.\,4} = \binom{9}{4}.$$

In selecting 4 things from 9 things of which one thing will be called A, we either select A or we do not select A. In the first case the number of selections (choosing 3 from the remaining 8) is $\binom{8}{3}$. In the second case we choose 4 from the remaining 8 in $\binom{8}{4}$ ways. $\therefore \binom{9}{4} = \binom{8}{3} + \binom{8}{4}$.

Permutations, New Usage, Block-Perm.

Suppose a choice of nine football matches is to be made from twelve. This can be done in ${}_{12}C_3$ or 220 ways.

Denoting the twelve matches by the letters A to L and marking those chosen with an o, three possible choices are shown in the diagram (Fig. 43).

If now the matches are divided into two blocks, labelled P and Q, and " block-perm. Q with P " is added, then instead of *three* selections there are shown *nine* selections, since the meaning is that each of the three selections of four of the matches in block Q is to be taken with each of the three selections of five of the matches in block P.

A	o	o		
B	o		o	
C		o	o	
D	o	o	o	P
E		o	o	
F	o		o	
G	o	o		
H	o	o		
I		o	o	
J	o		o	Q
K	o	o	o	
L	o	o	o	

FIG. 43

This is one of the ways in which those who fill in forms for " the

Pools " are saved the trouble of writing out separately a large number of selections. It is a slightly new sense for " permutations ", though not far from the ordinary English meaning of the word " permute ".

Examples 74

1. Show that in general $\binom{n+1}{r}=\binom{n}{r}+\binom{n}{r-1}$.
2. Simplify $\binom{n}{r}+\binom{n}{r}\cdot\binom{n-1}{r-1}$.
3. Find the value of $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$.
4. Find the value of $\binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6}$.
5. Show that $\binom{\frac{1}{2}}{r}$ is positive if r is odd and negative if r is even.
6. What is the smallest value of r for which $\binom{9\cdot 5}{r}$ is negative?
7. Show that $\binom{1\cdot 5}{3}+\binom{0\cdot 5}{3}=0$.
8. If five matches are to be chosen from the first seven of twelve matches and four matches from the last five, show that instead of ${}_{12}C_9=220$ possibilities there are 105. Why cannot these be completely covered in 11 columns, block-permed?
9. Suppose that in the case of twelve football matches, five are to be chosen from the first six and four from the second six. In how many ways can this be done?

   Using 4 columns and adding, " block-perm those in first six with those in second six ", show that 8/45 of the possible selections are included.

Miscellaneous Examples in Combinations and Permutations

In problems on selections and arrangements there are often various restrictive conditions, and the formulae for ${}_nC_r$ and ${}_nP_r$ can frequently not be applied to give the answer directly, but only to help. Any examples worked below should be gone through carefully.

Examples 75

1. How many " words ", each of 4 consonants and 3 vowels with the vowels in the even places, can be made from the letters of the word " *equivocal* "? The " words " need not have any meaning.

[Solution. There are only 4 consonants, so they must all be used.
3 vowels can be chosen from 5 in ${}_5C_3 = {}_5C_2 = 10$ ways.
The 4 consonants can be arranged in order in $4!$ ways.
The 3 vowels can be arranged in order in $3!$ ways.
The required number is $10 \times 4! \times 3!$ ways $= 1{,}440$ ways.]

2. Repeat Example 1 if the given word is "*facetiously*", counting y as a vowel, and placing vowels in the odd places and consonants in the three even places.

3. Find the number of changes that can be rung on a peal of 7 bells, each bell to be used once, and the last bell being a definite one.

4. How many different signals can be sent with 5 flags displayed (i) 3 at a time, (ii) 4 at a time, (iii) 2 at a time, (iv) any number at a time?

5. How many straight lines are obtained by joining in all possible ways 4 points of which no 3 are in the same straight line?

How many points of intersection are there of 4 straight lines, no 2 of which are parallel, and no 3 of which meet in a point?

6. In a telegraphic code there are two signs, a dot and a dash. How many letters can be made with these signs 1, 2, 3, or 4 at a time?

7. In how many ways can 5 ladies and 5 gentlemen sit at a round table, so that no two ladies sit together?

If the positions of the host and hostess are fixed, in how many ways can the party sit?

[At the round table, it is supposed that the position where the first person sits does not matter ; a general shift one place to the left, for example, would not change the order.]

8. In how many ways can 20 books be arranged on a shelf, so that a particular pair of books shall (i) come together in a particular order, (ii) not come together?

9. There are 4 letters and 4 directed envelopes. In how many ways can the letters be put into the envelopes, so that each is in a wrong one?

[To generalise this to n letters and n envelopes is a difficult problem, needing an elaborate method for its solution.]

10. Find the number of permutations three at a time of the letters in the words (i) PRESS ; (ii) ENVELOPE ; (iii) SWEETMEAT.

[Solution. (i) Consider the letters P, R, E, S.

These may be arranged three at a time in $4 \times 3 \times 2$ ways, i.e. 24 ways. The arrangements in which the two letters S appear have still to be counted. With the two letters S any one of the three others may be taken. This gives 3 such selections. Each of these 3 selections gives rise to 3 arrangements, for the third letter chosen

may be placed in one of three places; first, between the two letters S, or last.

Thus there are 9 of these arrangements, which with the 24 in which not more than one S occurs make a total of 33 permutations.]

11. Show that the answer to 10 (iii) is the coefficient of x^3 in

$$\lfloor 3 \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)(1+x)^4.$$

Explain the connection.

[This is a step towards a very complicated general formula.]

12. Prove that ${}_7C_4={}_5C_4+2\,.\,{}_5C_3+{}_5C_2$ without using the formulae.

[Solution. Suppose 7 letters a_1, a_2, b_1, b_2, b_3, b_4, b_5.

Then ${}_7C_4$ is the number of ways of choosing 4 letters.

But the three terms of the right-hand side represent

(i) the number of ways of choosing from the b's only;

(ii) the number of choices if one a is taken (in one of two ways) and the other 3 letters chosen are b's;

(iii) the number of choices if both a's are taken (in one way) and 2 of the b's.

Hence the result follows.]

13. As in No. 12, prove that (i) ${}_7C_4={}_6C_4+{}_6C_3$;

(ii) ${}_9C_5={}_6C_5+{}_3C_1\,.\,{}_6C_4+{}_3C_2\,.\,{}_6C_3+{}_6C_2$;

(iii) ${}_{10}C_4={}_7C_4+{}_3C_1\,.\,{}_7C_3+{}_3C_2\,.\,{}_7C_2+{}_3C_3\,.\,{}_7C_1$.

[To generalise Nos. 12, 13 gives an important theorem known as Vandermonde's theorem.]

14. How many different terms each of the seventh degree can be made with the three letters a, b, c?

[Solution. Consider three such terms $a^3b^2c^2$, a^2b^4c, and b^5c^2. Written in full these are $aaabbcc$, $aabbbbc$ and $bbbbbcc$.

These can be denoted by xxx|xx|xx, xx|xxxx|x and |xxxxx|xx. It will be seen that in each term there are 7 crosses and 2 lines and the number of such terms is the number of ways we can arrange 7 crosses and 2 lines, which has been shown (p. 179) to be

$$9!\div(7!\,2!).]$$

15. The general case corresponding to Ex. 14 is to find the number of products each of degree r to be made from n letters—these are usually called the *homogeneous products of degree r* and their number is denoted by ${}_nH_r$. Show by similar reasoning to that of Example 14, either in another special case or if possible in the general case that

$${}_nH_r=(n+r-1)!\div\{r!\,(n-1)!\}.$$

Miscellaneous Examples 76

1. A rectangular dormitory with a door at the middle of each long side contains 16 beds, 8 along each long side. In how many ways may 16 boys occupy the beds (i) if any boy may occupy any bed, (ii) if 2 of the boys must not sleep next to a door?

2. In how many ways can 5 rabbits run to earth if there are 10 holes that can be used, the order in which they run down the holes not being taken into account?

3. Find the number of ways in which the six letters, 2 N's, 2 O's, L, D can be arranged so as *not* to spell LONDON.

4. Twenty cricketers on tour consist of 2 wicket-keepers, 6 bowlers, and 12 batsmen. How many different teams of 11 players can be chosen if there must be at least one wicket-keeper and four bowlers in each team?

5. (i) How many different 5-figure numbers can be made with the figures 0, 1, 2, 3, 4, 5, 6, 7 (0 must not occupy the first place and each figure may only be used once)?

   (ii) With the figures used as in (i), how many of the five-figure numbers are multiples of 5?

6. N boys in a class each hand in an exercise book. In how many ways can the books be placed in a pile if two boys insist that neither of their books should be on top of the pile?

7. If ${}^{20}P_r : {}^{20}P_{r-2} = 132 : 1$, find r.

8. How many closed paths of six sides can be formed by joining up the six points A, B, C, D, E, F?

9. If there are n lines in a plane with no two lines parallel and no three lines concurrent, find the number of points of intersection.

10. How many different sub-committees of 3 men and 3 women can be formed from a full committee of 10 men and 8 women?

11. In how many ways may 14 threepenny pieces and 10 pennies be divided amongst four different people so that each receives the same amount? (N.)

12. The results of 21 football matches were to be predicted, and any side could win, lose or draw. How many different predictions could have been made in which exactly 18 results were given correctly? (N.)

13. If a diagonal of a polygon is defined as a line joining any two non-adjacent vertices, how many diagonals has a convex polygon of n sides?

    If no four vertices of the polygon are concyclic and $n \geqslant 6$, find how many circles can be drawn so that each passes through three vertices of the polygon, no two of which are adjacent. (N.)

14. (*a*) In how many ways can the letters a, b, c, d, p, q be placed in a row beginning with a, b, c or d and ending with p or q, each letter being used once only?

(*b*) Given two sets of letters :

$$a, b, c, d, e;$$
$$p, q, r, s, t,$$

find in how many ways a row, beginning with a letter from the first set and ending with a letter from the second set, can be made by taking all possible selections of seven letters, containing four from the first set and three from the second set. (N.)

15. State and prove a formula for the number of permutations of n things, taken all at a time, of which p are alike of one kind, q are alike of another kind, and the rest are all different.

In how many ways can nine balls, of which four are red, four are white and one black, be arranged in a straight line so that no red ball is next to the black?

In how many ways can the same nine balls be arranged so that no red ball is next to a white? (N.)

16. Prove that the number of permutations of n different things taken r at a time is $n!/(n-r)!$. If all are taken together, but n_1 are alike, another n_2 are alike, and so on, what is the number of permutations, and why?

Show that 6 black and 4 white balls can be arranged in 210 different ways.

The Binomial Theorem

The binomial theorem provides a rule for writing down the expanded form of $(x+a)^n$ or of $(1+x)^n$.

At first the restriction is made that *n is a positive integer*. It has been shown on p. 100 that in this case the coefficients can be found from the table known as Pascal's Triangle. In this method it is necessary to work up from the start, so that, to obtain the coefficients for the case $n=12$, one must first find those for all the cases given by $n=1, 2, 3$ ... up to 11.

It will now be explained how the coefficients for any positive integral value of n can be found without reference to those for smaller values of n. This needs the formula for ${}_nC_r$.

Binomial Theorem (n a positive integer)

In forming the expansion of $(x+a)(x+a)(x+a)$ each term is got by taking an x or an a from each bracket and multiplying them

together ; if it is decided to take an x from one of the brackets and the a from the remaining two brackets, a terms xa^2 is obtained, and such a term can be obtained as many times as it is possible to choose the bracket from which to take the x ; i.e. xa^2 can be obtained 3 times, and so the term $3xa^2$ appears in the expansion of $(x+a)^3$.

In the general case, the expansion of $(x+a)^n$:

(i) x taken from each bracket gives the term x^n.

(ii) x taken from $(n-1)$ brackets and a taken from the other bracket gives the term $x^{n-1}a$. This term will be obtained as many times as it is possible to choose the one bracket from which to take the a ; i.e. the term $x^{n-1}a$ is obtained ${}_nC_1$ times to give ${}_nC_1x^{n-1}a$.

x taken from $(n-2)$ brackets and a taken from the remaining 2 brackets gives the term $x^{n-2}a^2$; the two brackets may be chosen from the n brackets in ${}_nC_2$ ways, and so the term ${}_nC_2x^{n-2}a^2$ appears.

In the same way, x taken from $(n-r)$ brackets and a taken from the remaining r brackets gives the term $x^{n-r}a^r$; the r brackets from which to take the a may be chosen in ${}_nC_r$ ways, and so the term ${}_nC_rx^{n-r}a^r$ appears.

(iii) a taken from each bracket gives the term a^n.

Hence

$$(x+a)^n = x^n + {}_nC_1x^{n-1}a + {}_nC_2x^{n-2}a^2 + \dots + {}_nC_rx^{n-r}a^r + \dots + a^n. \qquad \dots\dots\dots(A)$$

Result (A) is called the Binomial Theorem for a *positive integral* index (i.e. n is a positive whole number) and the right-hand side is called the Binomial expansion of $(x+a)^n$.

Notice that the coefficients in the expansion are symmetrical since ${}_nC_r = {}_nC_{n-r}$.

The student should verify in one or two cases that the coefficients given in the Pascal Triangle of numbers agrees with (A).

Various other expansions are easily deduced ; for example,

$$\begin{aligned}(a-2b)^5 &= a^5 + {}_5C_1a^4(-2b) + {}_5C_2a^3(-2b)^2 + {}_5C_3a^2(-2b)^3 \\ &\quad + {}_5C_4a(-2b)^4 + (-2b)^5 \\ &= a^5 - 10a^4b + 40a^3b^2 - 80a^2b^3 + 80ab^4 - 32b^5.\end{aligned}$$

The minus sign of $-2b$ makes every second coefficient minus.

The term in x^5 in the expansion of $\left(x+\frac{1}{x}\right)^9$ must be obtained from $x^7 \times \left(\frac{1}{x}\right)^2$, and so its coefficient will be ${}_9C_2$, this being the number of ways of choosing two brackets from nine brackets.

If in the identity $(x+a)^{n+1}=(x+a)(x+a)^n$ we expand both sides and equate coefficients of x^r, we see that ${}_{n+1}C_r={}_nC_r+{}_nC_{r-1}$. This is, in fact, the rule by which the Pascal Triangle was obtained.

Examples 77

1. Write out the expansions of each of the following:
   (i) $(a-b)^6$; (ii) $(x-2)^6+(x+2)^6$; (iii) $(2a-3b)^5$.
2. Write out and simplify:
   (i) $(2x-y)^4$; (ii) $\left(x+\frac{3}{x}\right)^4-\left(x-\frac{3}{x}\right)^4$; (iii) $\left(x^2+\frac{1}{x}\right)^4$.
3. Find the term in x^3 in the expansions of:
   (i) $(x+a)^{12}$; (ii) $(x-2)^{10}$; (iii) $(2x-3a)^7$;
   (iv) $\left(\frac{1}{3}-3x\right)^5$; (v) $(3-x)^9$; (vi) $\left(2x^2+\frac{1}{x}\right)^9$.
4. Find the term in x^4 in the expansions of:
   (i) $(1+x)^2(2x+1)^5$; (ii) $(1-x)^2(x+2)^6$;
   (iii) $(2+3x)(1+x)^7$; (iv) $(3-x)(1+2x)^9$.
   [Solution (i). The required term is found from
   $(1+2x+x^2)(\ldots+{}_5C_1(2x)^4+{}_5C_2(2x)^3 \,.\, 1^2+{}_5C_3(2x)^2 \,.\, 1^3+\ldots)$.
   $\therefore$ The term is $({}_5C_1 \,.\, 2^4+2 \,.\, {}_5C_2 \,.\, 2^3+{}_5C_3 \,.\, 2^2)x^4$
   $=(5 \,.\, 16+2 \,.\, 10 \,.\, 8+10 \,.\, 4)x^4=280x^4$.]
5. Write down and simplify the coefficient of t^5 in $(2x+3ty)^7$.
6. Find the coefficient of x^7 in
   (i) $(a^2-2x^2)^5(1+x)$; (ii) $(1-ax)(1+a^2x^2)^5$.
7. Find the coefficient of x^4 in
   (i) $(1+x)^3(1-x)^3$; (ii) $\left(x+\frac{1}{x}\right)^2(1+x)^6$.

   Write down and simplify:
8. The 5th term of $\left(2x+\frac{b}{3}\right)^7$.
9. The middle term of $\left(\frac{x}{2}+y\right)^{12}$.
10. The middle term of $\left(\frac{3x}{4a}-\frac{4a}{3x}\right)^8$.
11. The term independent of x in $\left(x^2-\frac{1}{2x}\right)^{18}$.
12. The expansion of $(x+\sqrt{2})^6+(x-\sqrt{2})^6$.

13. Find the first four terms in the expansion of $(1+x+x^2)^n$.

[Solution. $$\{1+(x+x^2)\}^n = 1+n(x+x^2)+\frac{n(n-1)}{2}(x+x^2)^2 + \frac{n(n-1)(n-2)}{6}(x+x^2)^3+\ldots,$$

which as far as the term in x^3

$$= 1+nx+\{n+\tfrac{1}{2}n(n-1)\}x^2+\{n(n-1)+\tfrac{1}{6}n(n-1)(n-2)\}x^3+\ldots$$
$$= 1+nx+\tfrac{1}{2}(n^2+n)x^2+\tfrac{1}{6}n(n-1)(n+4)x^3+\ldots.]$$

14. Prove that the coefficients of x^2 and x^3 in the expansion of

$$(2+2x+x^2)^n$$

in ascending powers of x are $n^2 2^{n-1}$ and $\frac{1}{3}n(n^2-1)2^{n-1}$. (L.)

15. Find the positive integral value of n which makes the ratio of the coefficient of x^4 to that of x^3 in the expansion of $(1+2x+3x^2)^n$ in a series of powers of x equal to $121/28$. (L.)

16. By giving special values to x in the expansion of $(1+x)^n$, prove that

(i) $1+{}^nC_1+{}^nC_2+\ldots+{}^nC_{n-1}+1=2^n$;
(ii) $1-{}^nC_1+{}^nC_2-{}^nC_3+\ldots=0$;
(iii) $1-2\,.\,{}^nC_1+4\,.\,{}^nC_2-8\,.\,{}^nC_3+\ldots=(-1)^n$.

17. Find the sum of the coefficients in the expansions of

(i) $(x-y)^{12}$; (ii) $(x+2y)^5$; (iii) $(3x-2y)^{15}$.

Ratio of one term to the previous one

In $(1+x)^n$ the terms in x^{r-1} and x^r, which are the rth and $(r+1)$th terms, are

$$_nC_{r-1}x^{r-1} \text{ and } {}_nC_r x^r$$

or
$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{1\,.\,2\,.\,3\ldots(r-1)}x^{r-1}$$

and
$$\frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{1\,.\,2\,.\,3\ldots(r-1)(r)}x^r.$$

The second of these terms is got from the first by multiplying by

$$\frac{n-r+1}{r}x.$$

This is the ratio of the $(r+1)$th term to the rth term ; for example, in $\left(1+\frac{1}{5}\right)^{25}$ the ratio is $\frac{26-r}{r}\,.\,\frac{1}{5}$, so that the ratio of the 13th term to the 12th is $\frac{14}{12}\times\frac{1}{5}=\frac{7}{30}$.

Examples 78

1. In the expansion of $\left(1+\frac{1}{3}\right)^{20}$ write down the ratio of the $(n+1)$th term to the nth, and show that it is less than 1 if $n>5$.

2. Write out the expansion of $\left(1+\frac{1}{5}\right)^{18}$ until the greatest term is reached.

3. Find the greatest term in the expansions of

$$\text{(i) } (1+\cdot 3)^{20}; \quad \text{(ii) } (1+\cdot 07)^{50}.$$

4. Find which is the greatest term in the expansion of $(4+3x)^{12}$ when $x=\frac{2}{3}$. (L.)

5. Find (i) the greatest numerical factor of any term in the expansion of $(3a+5b)^{18}$, and (ii) the greatest term in the expansion when $a=\frac{1}{3}$, $b=\frac{4}{5}$. (L.)

6. Which is the numerically greatest term in

(i) $(5-3x)^7$ if $x=2$; (ii) $(7-2x)^6$ if $x=\frac{2}{3}$; (iii) $(1-2x)^7$ if $x=\frac{1}{8}$?

[The minus signs in these expressions have no effect on the answers.]

7. Which is the largest coefficient in the expansions of

$$\text{(i) } \left(2x+\frac{1}{3x}\right)^{20}; \quad \text{(ii) } (1+x)^{2n}.$$

8. Which term in the expansion of $(1+x)^{2n}$ has the largest coefficient? Which is the largest term if $x=\frac{2}{3}$ and $n=10$?

9. The expression $(3+5)^{20}$ is expanded by the Binomial Theorem; without writing down the expansion, prove that each of the 2nd, 3rd, ... 10th terms is more than double the term which precedes it, and that each of the 11th, 12th, ... 21st terms is less than double the term which precedes it. (N.)

10. In the expansion of $\left(1+\frac{x}{3}\right)^{13}$ find between what values x must lie in order that the seventh may be the greatest term. (B.)

11. Use the results

$$\begin{aligned}(1+x)^n &= (1+x)(1+x)^{n-1}\\ &= (1+x)^2(1+x)^{n-2}\\ &= (1+x)^3(1+x)^{n-3},\end{aligned}$$

by equating coefficients of x^r, to prove that

(i) ${}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1}$;

(ii) ${}_nC_r = {}_{n-2}C_r + 2 \cdot {}_{n-2}C_{r-1} + {}_{n-2}C_{r-2}$;

(iii) ${}_nC_r = {}_{n-3}C_r + 3 \cdot {}_{n-3}C_{r-1} + 3 \cdot {}_{n-3}C_{r-2} + {}_{n-3}C_{r-3}$.

12. Verify the results of No. 11 by use of the formulae for ${}_nC_r$, etc.

Binomial Expansion without notation for combinations

If the formulae obtained previously for ${}_nC_1$, ${}_nC_2$, etc., are written in full, the earlier terms of the binomial expansion read as follows :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \,.\, 2}x^2 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}x^4 + \ldots$$

or

$$1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \ldots .$$

In cases where the terms diminish rapidly, which will occur when x is small, or in the corresponding expansion for $(a+b)^n$ when $\frac{b}{a}$ is small, these early terms may be used to give an approximate value.

Thus $(1{\cdot}01)^{11} = \left(1 + \frac{1}{100}\right)^{11}$

$$= 1 + 11 \,.\, \frac{1}{100} + \frac{11 \times 10}{1 \,.\, 2} \,.\, \frac{1}{100^2} + \frac{11 \,.\, 10 \,.\, 9}{1 \,.\, 2 \,.\, 3} \,.\, \frac{1}{100^3} + \ldots .$$

The terms are diminishing rapidly ; the third is $55/10^4$ and the fourth is $165/10^6$, so the value of $(1{\cdot}01)^{11}$, found by the addition at the side, is $1{\cdot}1157$ to 4 decimal places.

$$\begin{array}{r} 1{\cdot}11 \\ {\cdot}0055 \\ {\cdot}000165 \\ \hline 1{\cdot}115665 \end{array}$$

Again, $(2{\cdot}08)^8 = 2^8(1 + {\cdot}04)^8$

$$= 2^8\left\{1 + 8 \times {\cdot}04 + \frac{8 \,.\, 7}{1 \,.\, 2} \times {\cdot}0016 + \frac{8 \,.\, 7 \,.\, 6}{1 \,.\, 2 \,.\, 3} \times {\cdot}000064 + \ldots\right\}$$

and in the { } the ratio of the 5th term to the 4th will be $\frac{5}{4} \times {\cdot}04 = \frac{1}{20}$.

$\therefore\ (2{\cdot}08)^8 \simeq 2^8\{1 + {\cdot}32 + 28 \times {\cdot}0016 + 56 \times {\cdot}000064\}$

$\simeq 2^8 \times 1{\cdot}3686 = 350{\cdot}4.$

$$\begin{array}{r} 1{\cdot}32 \\ {\cdot}0448 \\ {\cdot}00358 \\ {\cdot}00018 \\ \hline 1{\cdot}3686 \end{array}$$

4-figure logarithm tables give $350{\cdot}5$.

For the use of the Binomial Theorem in approximations, the important point is the *rapid* diminution of the terms. This occurs more often in practical applications than might be supposed, because it is so often needful to calculate the effect due to *small* changes in the data.

Examples 79

1. Use the first three terms of a binomial expansion to find approximations to

$$\text{(i) } 1{\cdot}01^9; \quad \text{(ii) } (1{\cdot}005)^{12}; \quad \text{(iii) } (2{\cdot}03)^7.$$

Compare with the results given by 4-figure tables.

2. Seven-figure tables give $1{\cdot}003^{10} \simeq 1{\cdot}030408$. Show that taking 4 terms of the binomial expansion for $(1 + {\cdot}003)^{10}$ gives this result. Find by how much the result of taking only 3 terms falls short.

3. The radius of a sphere is increased by ·2%. Show that the volume is increased by approximately ·6%, but actually by rather more than this.

4. Write down the first three terms in the expansion of $\left(1 + \frac{r}{5200}\right)^{52}$.

Compound interest is paid on a sum of money at the rate of $\frac{4{\cdot}9}{52}$%

per week. Show that this is very nearly equivalent to 5·02% per annum. (L.)

5. Apply the Binomial Theorem to evaluate

$\left(\frac{593}{600}\right)^{12}$ correct to six decimal places. (L.)

The Binomial Theorem for any index

If n is not a positive integer, the expression ${}_nC_r$ is undefined and meaningless ; but the Binomial Theorem, in the form

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \dots$$
$$+ \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \cdot 2 \cdot 3 \dots r}x^r + \dots \qquad \dots\dots\text{(I)}$$

is found to be true *if x is numerically less than* 1.

The proof is beyond the scope of this chapter, except in a few special cases, but the theorem will be used in examples. (For proof, see p. 228.)

If n is a positive integer it will be recognised that this expansion on the R.H.S. of (I) is $1 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_rx^r + \dots$ and will terminate with ${}^nC_nx^n$, i.e. with x^n, for subsequent terms will all be zero. (Putting $r = n+1$ in $n-r+1$ gives a zero coefficient to x^{n+1}.) On the other hand, putting $n = \frac{1}{2}$ or -3, for instance, in the R.H.S. of (I) gives a series which goes on and on ; i.e. contains an infinite number of terms, for no one of the coefficients will be zero. The coefficient of x^r is $\binom{n}{r}$.

Case of n a Negative Integer

The following are some simple cases for which n is a negative integer and which can be verified by elementary methods.

$$(1+x)^{-1}=1+\frac{(-1)}{1}\,x+\frac{(-1)(-2)}{1\,.\,2}\,x^2+\frac{(-1)(-2)(-3)}{1\,.\,2\,.\,3}\,x^3+\frac{(-1)(-2)(-3)(-4)}{1\,.\,2\,.\,3\,.\,4}\,x^4+\ldots$$

$$=1-x+x^2-x^3+x^4-\ldots\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$(1+x)^{-2}=1+\frac{(-2)}{1}\,.\,x+\frac{(-2)(-3)}{1\,.\,2}\,x^2+\frac{(-2)(-3)(-4)}{1\,.\,2\,.\,3}\,x^3+\frac{(-2)(-3)(-4)(-5)}{1\,.\,2\,.\,3\,.\,4}\,x^4+\ldots$$

$$=1-2x+3x^2-4x^3+5x^4-\ldots\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$(1+x)^{-3}=1+\frac{(-3)}{1}\,.\,x+\frac{(-3)(-4)}{1\,.\,2}\,x^2+\frac{(-3)(-4)(-5)}{1\,.\,2\,.\,3}\,x^3+\frac{(-3)(-4)(-5)(-6)}{1\,.\,2\,.\,3\,.\,4}\,x^4+\ldots$$

$$=1-3x+\frac{3\cdot4}{2}\,x^2-\frac{4\cdot5}{2}\,x^3+\frac{5\cdot6}{2}\,x^4-\ldots\;\ldots\ldots\ldots\text{(iii)}$$

Of these, the R.H.S. of (i) is a G.P., first term 1, common ratio $-x$, and therefore sum $1/(1-(-x))=(1+x)^{-1}$. The result is therefore true if $|x|<1$.

The result (ii) is given by the division below, which can be carried as far as is wished.

$$\begin{array}{r|l|l}
1+2x+x^2 & 1 & 1-2x+3x^2-4x^3+\ldots \\
 & \underline{1+2x+x^2} & \\
 & -2x-x^2 & \\
 & \underline{-2x-4x^2-2x^3} & \\
 & \qquad 3x^2+2x^3 & \\
 & \qquad \underline{3x^2+6x^3+3x^4} & \\
 & \qquad\quad -4x^3-3x^4 &
\end{array}$$

Note, however, that putting $x=2$ in $(1+x)^{-2}$ gives the result $1/9$, while putting $x=2$ in the expansion gives $1-4+12-32+\ldots$, a series of integers which cannot be equal to $1/9$. In fact, the result is *not true* unless $|x|<1$.

Case of n a Fraction

Suppose that $(1+x)^{\frac{1}{2}} = 1 + ax + bx^2 + cx^3 + \ldots$ and try to work out $a, b, c \ldots$.

Squaring $$1 + x = (1 + ax + bx^2 + cx^3 + \ldots)(1 + ax + bx^2 + cx^3 + \ldots)$$
$$= 1 + 2ax + (a^2 + 2b)x^2 + 2(ab + c)x^3 + \ldots$$

Equating coefficients,
$$2a = 1,$$
$$a^2 + 2b = 0,$$
$$2(ab + c) = 0,$$
etc.

These give $a = \frac{1}{2}$, $\frac{1}{4} + 2b = 0$; $\therefore\ b = -\frac{1}{8}$, $ab + c = 0$; $\therefore\ c = +\frac{1}{16}$, and this process can be continued till our patience is exhausted.

Now if the binomial expansion fits this case,

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \ldots.$$

This also gives $a = \frac{1}{2}$, $b = -\frac{1}{8}$, $c = \frac{1}{16}$, so the use of the binomial expansion seems to be justified.

Approximation

These cases of the Binomial Theorem when n is a fraction are valuable for rapid approximation if x is small.

For example, cube roots for numbers near unity will be given by

$$(1+x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}x^3 + \ldots$$
$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \ldots.$$

Suppose it is desired to calculate $\sqrt[3]{(1{\cdot}02)}$ using this last series. Disregarding sign, the multipliers giving the successive terms are $\frac{1}{3} \times \frac{1}{50}$, $\frac{1}{3} \cdot \frac{1}{50}$, $\frac{5}{9} \cdot \frac{1}{50}$ or $\frac{1}{90}$, $\frac{8}{12} \times \frac{1}{50}$ or $\frac{1}{75}$.

So the calculation goes thus:

3)	·020,000,0		
3)	·006,666,7	2nd term	1·006,666,7
50)	·002,222,2		+ ·000,000,5
90)	·000,044,4	3rd term	1·006,667,2
75)	·000,000,5	4th term	− ·000,044,4
	·000,000,0	5th term	1·006,622,8

Thus $\sqrt[3]{1{\cdot}02} = 1{\cdot}006{,}622{,}8$.

Using logarithms 5-figure tables give 1·006,7,
while 7-figure tables give 1·006,623.

Thus the binomial series gives the 7-figure result using 4 terms, and would give 2 more figures if the 5th term were used.

Example I. Find $\sqrt[4]{626}$ by the Binomial Theorem.

$$626 = 625 + 1 \text{ and } 625 = 5^4;$$

$$\therefore \sqrt[4]{626} = \sqrt[4]{(5^4+1)} = 5\left(1 + \frac{1}{5^4}\right)^{\frac{1}{4}}$$

$$= 5\left(1 + \frac{1}{4} \cdot \frac{1}{5^4} + \frac{\frac{1}{4}(\frac{1}{4}-1)}{1 \cdot 2} \cdot \frac{1}{5^8} + \dots\right).$$

To get the third term from the second multiply by

$$-3/8 \times 5^4 = -3/5000$$
$$= -6/10000.$$

The second term is $1/4 \times 5^3 = \dfrac{2}{1000} = \cdot002$.

The third term is $\cdot002 \times (-6/10000) = -\cdot0000012$.

$$\therefore \sqrt[4]{626} = \left\{\begin{array}{r} 5\cdot0020000 \\ -\ \cdot0000012 \end{array}\right. = 5\cdot0019988.$$

Seven-figure logarithm tables give 5·00200.

Examples 80

1. Write down and simplify the series for $(1-x)^{-1}$, $(1-x)^{-2}$, $(1-x)^{-3}$, and note that all the terms are positive.
2. Write down and simplify the series for $(1-x)^{\frac{1}{2}}$ to six terms, and note that after the first all the terms are negative.
3. Write down and simplify the first four terms in the expansions of (i) $(1+x)^{\frac{2}{3}}$; (ii) $(1-2x)^{-\frac{3}{2}}$.
4. Write down and simplify :
   (i) the 7th term of $(1-2x)^{\frac{4}{3}}$;
   (ii) the 6th term of $(1-x)^{-7}$.
5. Write down and simplify :
   (i) the rth term of $(1+x)^{\frac{5}{4}}$;
   (ii) the $(r+1)$th term of $(1-x)^{-r}$;
   (iii) the $(n+1)$th term of $(1-x)^{-\frac{p}{q}}$.
6. Find to three places of decimals, using the binomial expansion : (i) $\sqrt{101}$; (ii) $\sqrt[3]{1003}$; (iii) $\sqrt[6]{63}$.

7. Write down and simplify the first five terms in the expansion of $(1-\frac{2}{3})^{\frac{1}{3}}$, and hence show that the sum of the series

$$\frac{2}{9}+\frac{2}{2!}\left(\frac{2}{9}\right)^2+\frac{2\cdot5}{3!}\left(\frac{2}{9}\right)^3+\frac{2\cdot5\cdot8}{4!}\left(\frac{2}{9}\right)^4+\ldots$$

is $1-\sqrt[3]{\frac{1}{3}}$.

8. Use the Binomial Theorem to expand both of the expressions

$$\sqrt{(1-2x)},\quad \frac{2-3x}{2-x}$$

in ascending powers of x as far as the term in x^4 in each case; state the range of values of x for which each of the expansions is valid.

Prove also by putting $x=\frac{1}{50}$ in your expansions that $\sqrt{6}$ differs from $\frac{485}{198}$ by less than 0·00001. (O. & C.)

Application to Small Changes

To show that *the coefficient of cubical expansion for a given material is three times the coefficient of linear expansion* it is only necessary to to show that if a is increased by $x\%$ when x is small, then a^3 is increased by $3x\%$.

If a is changed to $a\left(1+\frac{x}{100}\right)$ then a^3 becomes $a^3\left(1+\frac{x}{100}\right)^3$, but

$$\left(1+\frac{x}{100}\right)^3=1+\frac{3x}{100}+\frac{3x^2}{100^2}+\frac{x^3}{100^3}\simeq 1+\frac{3x}{100}\text{ when } x \text{ is small}$$

$\therefore$ a^3 becomes $a^3\left(1+\frac{3x}{100}\right)$ and has been increased by $3x\%$.

Again, Kepler discovered the law that

"*The square of the periodic time of a planet is proportional to the cube of its average distance from the sun,*"

$$\text{or } T^2=kD^3 \text{ or } T=\sqrt{k}\,.\,D^{\frac{3}{2}}.$$

Suppose now that D were increased by a small percentage $x\%$ with the result that T was increased by $y\%$.

Then
$$T\left(1+\frac{y}{100}\right)=\sqrt{k}\,.\,D^{\frac{3}{2}}\left(1+\frac{x}{100}\right)^{\frac{3}{2}}$$
$$\simeq\sqrt{k}\,.\,D^{\frac{3}{2}}\left(1+\frac{3x}{200}\right)\text{ if } x \text{ is small}$$
$$\simeq T\left(1+\frac{3x}{200}\right).$$

Hence the percentage increase in T is approximately $\frac{3}{2}x$, one-and-a-half times the percentage increase in D.

Examples 81

1. The time of the to-and-fro swing of a pendulum of length x inches is $2\pi\sqrt{(x/386)}$ seconds. Find the percentage increase in the time if x is increased by $\cdot 2\%$ (i.e. if x is replaced by $x \times 1\cdot 002$).

   Find also the length of the "seconds" pendulum (such that the above time is 2 seconds) and how much a clock loses in the day if the "seconds" pendulum expands by $\cdot 1$ inch.

2. If in a fraction $\dfrac{p}{q}$ the denominator q is increased by $\cdot 1\%$ show that the fraction is decreased by $\cdot 1\%$ approximately, but that the decrease is somewhat less than $\cdot 1\%$ if calculated more accurately.

3. If in a fraction $\dfrac{p}{q}$ the denominator q is decreased by $\cdot 1\%$ show that the fraction is increased by $\cdot 1\%$ approximately, but that the increase is somewhat more than $\cdot 1\%$ if calculated more accurately.

4. If the area of a square is increased by a small percentage such as $\cdot 3$, show that its side is increased by approximately half this percentage.

   Is this approximation in excess or in defect?

5. The speed of a train is decreased by $\cdot 2\%$. What effect does this have on the time taken for a given distance?

6. A given length of ground is to be measured out with a yard measure. Owing to heat the measure expands by $\cdot 2$ inch. By how much % will this cause the estimate formed of the length to be incorrect?

7. Show that the saving in the time of a given journey due to a slight increase in the speed is to a first approximation the same as the loss in time due to an equal decrease, but that to a second approximation the latter is the greater.

Probability

Nearly everyone knows what is meant by saying that "the odds are two to one against" a certain event, or what is the same thing "the *chance* or *probability* of the event is one in three"; but it is less easy to give an exact definition of probability.

There are two definitions commonly given; to both there are theoretical objections, but both work out well enough in practice, and both lead to the same methods in attacking problems involving chances or probabilities.

One method *implies the knowledge of a large number of similar events.*

Suppose that we know that in a certain town only 4 days of May —taking the average of many years—are rainless; what is the chance that May 15th next year will be rainless?

The answer would be that the chance of no rain is 4/31.

Again, suppose a "mortality" table has the following entries dealing with 100,000 persons alive at the age of 10:

Age - - -	10	40	50
No. of survivors	100,000	82,284	72,726

These entries show that the chance of a person of age 40 being alive 10 years later is $72,726 \div 82,284 \simeq \cdot 884$.

It is easy to see that this method of estimating probabilities is not perfect. In the first example, more accurate study of the records might show that the middle of May is less rainy than the beginning or end. In the second example, the 82,284 would include men who had joined different trades, some of which would be less healthy than others.

Both solutions imply the idea of "*equally likely*"; in the first it has been tacitly assumed that rain is equally likely on one day of May as on another; in the second that, of the 82,284 persons, one is as likely to die as any other.

The other definition of probability *starts* with the idea of equally likely, and the natural examples are from games of chance, e.g. tossing coins or dice or drawing cards from a pack. Thus if a penny is tossed, we assume it is equally likely to fall "head" or "tail", and the chance of a "head" is one-half.

If a die in the form of a cube is not loaded in any way, then if it is thrown, any one of the 6 faces should be equally likely to end uppermost, so the chance or probability of throwing a six is 1/6; the chance of throwing a six or a five is 2/6; the chance of not throwing a six is 5/6.

A weak spot in this way of starting is that it assumes a knowledge that six possible results have *equal probabilities* in order to explain what probability means.

However, this definition is usually given:

If an event can happen in x ways and fail in y ways and all these ways are *equally likely* to occur, then the *probability* or *chance* of the event happening is $\frac{x}{x+y}$ and of its failing is $\frac{y}{x+y}$.

The other way of looking at it gives this definition:

If, on taking a very large number N of a series of cases in which an event A is in question, A happens on M occasions, the probability of the event A is $\frac{M}{N}$.

That the two definitions come to much the same thing in practice will be seen by considering that if a "die" were tossed 600,000 times, the number of sixes tossed would be *approximately* 100,000, since there is no reason why one face should appear more often than another giving the chances as 1/6 as before. It is, of course, extremely unlikely that the number would be *exactly* 100,000; on the other hand, if the number was completely different from this, say 150,000, the immediate inference would be that the die was untrue and weighted so as to be more likely to fall at "six" than it ought to do.

Suppose now that 9 pennies are tossed; what is the chance of there being 6 heads and 3 tails?

This sort of question is answered by thinking about combinations.

The total number of different possibilities is 2^9, for it is 2 for each of the 9 coins, i.e. is 512.

In how many of these cases are there 6 heads and 3 tails?

The choice of the three pennies to come "tails" can be made in ${}_9C_3$ ways, and for each of these ways there is exactly one arrangement of the heads and tails. So the number of cases of "success"—of the 6 heads and 3 tails appearing—is ${}_9C_3=\dfrac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}=3\times 4\times 7=84$.

The chance therefore is $84/512=21/128$, or just under one-sixth.

Card games are fertile in such questions of chance; for instance: the hearts in a pack of cards are separated and shuffled and four of the 13 cards are drawn; what is the chance that the king is among the four?

Four cards can be drawn from 13 in ${}_{13}C_4$ ways *all supposed to be equally likely*. If the king is to be one of the four, the other three are chosen from 12, the number of ways being ${}_{12}C_3$.

The chance of success is therefore ${}_{12}C_3\div{}_{13}C_4$, which

$$=\frac{12\cdot 11\cdot 10}{1\cdot 2\cdot 3}\times\frac{1\cdot 2\cdot 3\cdot 4}{13\cdot 12\cdot 11\cdot 10}=\frac{4}{13}\,.$$

Examples 82

1. Show that with 9 pennies the chance of 5 heads and 4 tails is 63/256 or just under one quarter.
2. Find the chance of 5 heads and 2 tails if 7 pennies are tossed.
3. (i) What is the chance of throwing 2 sixes with 2 dice?
   (ii) What is the chance of throwing exactly 2 sixes with 3 dice? with n dice?
4. From the 13 hearts, 5 cards are drawn; what is the chance of drawing both king and knave?

5. From the full pack of 52 cards, 7 cards are drawn; what is the chance of the ace of spades being one of them?

6. A bag contains 7 apples and 6 oranges. Two fruits are drawn at random. Show that the chance that 2 apples are drawn is ${}_7C_2 \div {}_{13}C_2$ and evaluate this.

7. Show that the chance of throwing 10 with 2 dice is one-twelfth. Explain why the chance of throwing 7 is double this.

8. In forecasting the results of 10 football matches, find the chance of (i) having all correct, (ii) having 8 correct.

9. Two boys each throw 2 dice. If the first scores 10, what is the chance the second will not score less?

10. Two brothers and two sisters enter for a tennis tournament for mixed doubles for which partners are drawn. Find the chance that neither brother is partnered with a sister if there are 32 couples.

Compounding Probabilities

(i) If two events are *mutually exclusive*, the chance of *one or other* happening is the *sum* of their probabilities.

Thus in tossing a die the chance of a six is $\frac{1}{6}$ and of a five is $\frac{1}{6}$.

Hence the chance of tossing either a five or a six is $\frac{2}{6}$.

(ii) If two events are *independent* the chance that they *both* happen is the *product* of their probabilities.

For example, if two dice are tossed and the question is asked, what is the chance of a six on the first and a five on the second the answer is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$; for whatever number occurs on the first die there are 6 possibilities for the number on the second to go with it. So there are 36 equally likely arrangements of which one is that required.

This last case is a good example of the care needed in the *wording* of questions on probability.

"What is the chance of throwing a five and a six in a throw with two dice"? is a *different* question, for the five may be on the first die, so there are *two* of the 36 arrangements which give five-and-six, and the chance is $\frac{2}{36}$ or $\frac{1}{18}$.

Another instance is when a pack of 52 cards containing 4 kings is shuffled. The chance that a card drawn from the pack would be a king is $\frac{4}{52}$, i.e. $\frac{1}{13}$. If a king were drawn and not replaced, the chance that the next card drawn would be a king is $\frac{3}{51}$, since of the possible results of the second draw only 3 would be kings.

The chance that the two draws would give kings is $\frac{4}{52} \times \frac{3}{51}$. This

result can also be arrived at by recognising that the 52 cards can be arranged in pairs in 52×51 ways, while the 4 kings can be arranged in pairs in 4×3 ways ; hence the chance of drawing 2 kings is $\dfrac{4 \times 3}{52 \times 51}$.

Examples 83

1. A die has its faces numbered 1 to 6. What is the chance of throwing (i) less than 3? (ii) more than 3?
2. What is the chance of drawing at least one ace when two cards are drawn from a pack of 52 cards containing 4 aces (i) when the card first drawn is replaced before the second draw? (ii) when the card first drawn is not replaced?

   [In (ii) consider the number of arrangements of the cards in pairs and the number of these containing 1 or 2 aces.]
3. A bag contains 5 white balls and 7 red balls all the same size. If two balls are drawn from the bag, find the probability that they will be (i) both white, (ii) both red, (iii) one red and one white.
4. Two people are each asked to write down a number less than 20 (0 not allowed). What is the chance that the numbers (i) are both less than 10? (ii) are the same? (iii) add up to 10?
5. What is the chance that a three-figure car number will contain (i) at least one figure seven? (ii) at least 2 sevens?
6. A and B toss a coin alternately. Show that the chance A tosses the first " head " is 2/3. [Hint : the chance that A tosses a head at the first, second, third attempts, etc., are terms of a G.P.]

Using the Binomial Expansion to calulate Probabilities

Certain types of probabilities are best obtained by making use of the binomial expansion, and are illustrated by the following examples.

Example I. Find the chance of throwing a six three times in eight throws of a die.

The chance of throwing a six in any one throw is $\frac{1}{6}$, while the chance of *not* throwing a six is $\frac{5}{6}$.

If the order in which the sixes are to be thrown is prescribed, then the chance of throwing the three sixes in the particular throws is $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$, while the chance of *not* throwing a six in the other throws is

$$\tfrac{5}{6} \times \tfrac{5}{6} \times \tfrac{5}{6} \times \tfrac{5}{6} \times \tfrac{5}{6},$$

i.e. the chance for a particular order for the three sixes is $(\frac{1}{6})^3 \times (\frac{5}{6})^5$.

Since it does not matter which three throws give sixes, their positions may be any of the ${}^8C_3$ ways of choosing the three throws to give sixes from the eight throws.

$\therefore$ the chance of throwing three sixes is ${}^8C_3(\frac{1}{6})^3(\frac{5}{6})^5$.

This chance will be recognised as a term in the binomial expansion of $(\frac{1}{6}+\frac{5}{6})^8$, and it can be shown that each term in the expansion gives the probability for a similar occurrence.

For instance, the chance of throwing a six four times is ${}^8C_4(\frac{1}{6})^4(\frac{5}{6})^4$, while the chance of not throwing a six is $(\frac{5}{6})^8$.

Example II. Find the chance of throwing at least five sixes in seven throws of a die.

As in Example I, the chance of throwing five sixes is ${}^7C_5(\frac{1}{6})^5(\frac{5}{6})^2$.

Similarly the chance of throwing six sixes is ${}^7C_6(\frac{1}{6})^6(\frac{5}{6})$,

and the chance of throwing seven sixes is $(\frac{1}{6})^7$.

The chance of throwing at least five sixes is the sum of the above chances, and so the required chance is

$$\frac{1}{6^7}\left\{\frac{7\cdot 6}{1\cdot 2}\cdot 25+7\cdot 5+1\right\}=\frac{561}{6^7}=\frac{187}{93312}\simeq\frac{1}{499}.$$

Example III. Three dice are thrown at the same time; find the probability that the sum of the numbers thrown is 10.

Here it is necessary to find out how many arrangements of the numbers 1, 2, 3, 4, 5, 6 three at a time will give 10 for their sum when any number may be repeated.

It will be seen that the coefficient of x^{10} in the expansion of

$$(x^1+x^2+x^3+x^4+x^5+x^6)^3$$

is the required number of arrangements.

The coefficient of x^{10} in $x^3(1+x+x^2+x^3+x^4+x^5)^3$ is also the coefficient of x^{10} in $x^3\left(\frac{1-x^6}{1-x}\right)^3$,

i.e. in $x^3(1-3x^6\ \dots)(1-x)^{-3}$,

i.e. in $x^3(1-3x^6\ \dots)\left(1+3x+\frac{3\cdot 4}{1\cdot 2}x^2+\frac{4\cdot 5}{1\cdot 2}x^3+\ \dots\ \frac{8\cdot 9}{1\cdot 2}x^7\ \dots\right)$

$\therefore$ the coefficient of x^{10} is $\frac{8\cdot 9}{1\cdot 2}-3\cdot 3=27$.

Also the number of possible arrangements of the scores on the dice is $6^3=216$

$\therefore$ the probability of throwing a total score of 10 is $\frac{27}{216}$, i.e. $\frac{1}{8}$.

Examples 84

1. Two dice are thrown at the same time; find the chance that the total score is five. [Use method of Example III above, and also from first ideas of probability.]

2. Four people are each asked to write down a number less than 10 (zero being excluded). What is the chance that the sum of the four numbers should be (i) 7? (ii) 16?

3. A and B throw a die in turn. Find the chance A throws the first six.

4. In eight throws of a die, what is the chance of throwing five five times?

5. Two numbers less than 10 are chosen at random (zero being excluded); what is the chance that their product is greater than 25?

6. What is the chance of three "heads" being obtained when 4 coins are tossed?

7. When three dice are thrown, what is the chance of the total score being 8?

8. If in throwing a die, to throw 5 or 6 is counted "success" and throwing 1, 2, 3, or 4 is counted "failure", show that continuous "failure" in 5 throws is 32 times as likely as continuous success, while the chance of three or more "failures" is to the chance of three or more "successes" as 192 : 51.

Miscellaneous Examples 85

1. (i) If ${}_nP_5 \div {}_nP_3 = 20$, find n.
(ii) Prove that ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$.
(iii) Prove that ${}_nC_r = {}_{n-2}C_r + 2\,{}_{n-2}C_{r-1} + {}_{n-2}C_{r-2}$.

2. Find the number of permutations (a) two at a time, (b) four at a time, of the letters in the words (i) SHINGLE; (ii) PASTRY; (iii) GERANIUM.

3. A rugby side consists of a scrum of eight and seven outsides. If each member of the scrum can play in any of the eight positions and each outside can play in any outside position, how many different arrangements are possible?

If in addition two members of the scrum can play in any of the seven outside positions while two outsides can play in any scrum position, how many different arrangements are possible?

4. A dramatic club has 15 women members and 12 men members. How many different casts of 5 women and 3 men can be chosen? (Any way of choosing the men is possible with each way of choosing the women.)

5. Find the coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$. If this coefficient is equal to that of x^{-7} in the expansion of $\left(ax + \frac{1}{bx^2}\right)^{11}$, prove that $ab = 1$. (L.)

6. Show that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms in the expansion of $(1+x)^{2n-1}$.
Find the coefficient of x in the expansion of

$$\left(2x-\frac{3}{x}\right)^9. \qquad \text{(L.)}$$

7. Show that the expansion of

$$\left(x^2-\frac{3}{x}\right)^n$$

in ascending powers of x contains a term independent of x if n is a multiple of 3, and evaluate this term (i) for $n=3$, (ii) for $n=12$. (L.)

8. (i) Show that the sum of the coefficients in the expansion of $(a+b)^6$ is double the sum for $(a+b)^5$, preferably without evaluating the coefficients.
(ii) In the expansion of $(1+x)^n$, what is the ratio of the coefficient of x^4 to the coefficient of x^3? (B.)

9. By considering the multiplication of $(1+x)^n$ by $(1+x)$ prove that ${}^{n+1}C_3={}^nC_3+{}^nC_2$.
Find the coefficient of the middle term in $(1+x)^n$ if $n=16$. (B.)

10. If n is a positive integer, what is the fourth term in the expansion of $(a+b)^n$?
In the expansion of $(1+0{\cdot}1)^n$ the ratio of the fifth term to the fourth term is $0{\cdot}4$, what is the value of n? (B.)

11. (i) If

$$(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_rx^r+\ldots+c_nx^n,$$

and

$$(1+x)^{n-2}=b_0+b_1x+b_2x^2+\ldots+b_rx^r+\ldots+b_{n-2}x^{n-2},$$

prove that

$$c_r=b_{r-2}+2b_{r-1}+b_r.$$

(ii) Find the coefficient of x^8 in the expansion of

$$\left(2+\frac{1}{x}+x\right)^{10}. \qquad \text{(B.)}$$

12. Write down and simplify the first five terms in the binomial expansion for $(1+x)^{\frac{1}{3}}$ and also the ratio of the $(n+1)$th term to the nth term.
Find the sum to infinity of the series

$$\frac{2}{9}+\frac{2}{2!}\left(\frac{2}{9}\right)^2+\frac{2\cdot 5}{3!}\cdot\left(\frac{2}{9}\right)^3+\frac{2\cdot 5\cdot 8}{4!}\cdot\left(\frac{2}{9}\right)^4+\ldots. \qquad \text{(B.)}$$

13. If $(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$, show

(i) $c_0 + c_1 + c_2 \ldots + c_n = 2^n$. [Put $x = 1$.]

(ii) $c_0 - c_1 + c_2 \ldots + (-1)^n c_n = 0$. [Put $x = -1$.]

(iii) $c_0 + c_2 + c_4 + \ldots = c_1 + c_3 + c_5 + \ldots$.

(iv) $c_1 + 2c_2 + 3c_3 + \ldots + nc_n = n \cdot 2^{n-1}$. [Differentiate.]

(v) $c_0 + 2c_1 + 3c_2 + \ldots + (n+1)c_n = (n+2^n) \cdot 2^{n-1}$.

(vi) $\dfrac{c_0}{1} + \dfrac{c_1}{2} + \dfrac{c_2}{3} + \ldots + \dfrac{c_n}{n+1} = (2^{n+1} - 1)/(n+1)$.

[Use integration.]

(vii) $c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2 = (2n)!/n!\, n!$.

[Pick out coefficient of x^n in $(1+x)^n \times (x+1)^n$.]

14. The letters of the word EASTER are jumbled, and then set out in a row. What is the chance they will again make the word EASTER?

15. A coin is tossed 6 times. Find the chance there will be three heads in succession.

16. Four players A, B, C, D throw a die in turn. Show that A's chance of throwing the first six is $6^3/(6^4 - 5^4)$.

CHAPTER IX

FINITE SERIES

Fundamental Equation

If u_n is a function of a positive integer n, the terms $u_1, u_2, u_3, \ldots$ form a *sequence*, and when these are added together we get the *series* $u_1+u_2+u_3+\ldots+u_n$, of which u_n is the nth term.

Taking S_n to be the sum to n terms, we have

$$S_n \quad = u_1+u_2+u_3+\ldots+u_{n-1}+u_n$$
$$\text{and } S_{n-1} = u_1+u_2+n_3+\ldots+u_{n-1},$$
$$\text{so that } S_n - S_{n-1} = u_n. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

This equation is fundamental in summation of series.

The equation (I) can be regarded as in (i) and (ii) below:

(i) Given the function S_n, then u_n and the series can be found.

e.g. To find the series for which $S_n = n^2$.

$$S_{n-1} = (n-1)^2 \text{ and } S_n - S_{n-1} = n^2 - (n-1)^2 = 2n-1;$$
$$\therefore\ u_n = 2n-1,\ u_1 = 2 \cdot 1 - 1 = 1,\ u_2 = 2 \cdot 2 - 1 = 3.$$

Therefore the series is $1+3+5+\ldots+(2n-1)+\ldots$.

(ii) Given the function u_n it is sometimes possible to find S_n.

e.g. To find the sum of the series for which $u_n = n$.

The identity $n(n+1)-(n-1)n \equiv 2n$ leads to

$$2u_n = 2n = n(n+1)-(n-1)n.$$

Writing $(n-1)$ for n: $2u_{n-1} = 2(n-1) = (n-1)n - (n-2)(n-1)$.

....

Writing r for n: $2u_r \quad = 2r = r(r+1)-(r-1)r$,

....

Writing 2 for r: $2u_2 \quad = 2 \cdot 2 = 2 \cdot 3 - 1 \cdot 2$.

Writing 1 for r: $2u_1 \quad = 2 \cdot 1 = 1 \cdot 2 - 0 \cdot 1$.

If all the n lines similar to the above were written down, the sum of the L.H. sides would be $2S_n$, while in summing the R.H. sides, pairs of terms cross out, e.g. 2nd term of first line with first term of second line, and only the $n(n+1)$ of the top line is left.

$$\therefore\ 2S_n = n(n+1), \text{ i.e. } S_n = \tfrac{1}{2}n(n+1)$$

The method of example (ii) depends on being able to express u_n as a difference in the form $f(n)-f(n-1)$ where in this case

$$f(n)=\tfrac{1}{2}n(n+1).$$

A set of important identities of this type, which should be noted, are:

$$\begin{aligned} n(n+1) &\equiv \tfrac{1}{3}n(n+1)(n+2) &&- \tfrac{1}{3}(n-1)(n)(n+1), \\ n(n+1)(n+2) &\equiv \tfrac{1}{4}n(n+1)(n+2)(n+3) &&- \tfrac{1}{4}(n-1)(n)(n+1)(n+2), \\ \text{etc.} \quad & \quad \text{etc.} && \quad \text{etc.} \end{aligned}$$

At this stage it is convenient to adopt another notation; $\sum\limits_1^n r$ is taken to denote the sum of all the terms obtained by giving r all the values from 1 to n, and thus $\sum\limits_1^n r=1+2+3+\ldots+r+\ldots+n$, so that, for example, $\sum\limits_1^{10} r=1+2+3+\ldots+9+10$.

Consequently the result proved in (ii) may be written

$$\sum_1^n r=\tfrac{1}{2}n(n+1).$$

The other two identities given above can be shown to lead to

$$\sum_1^n r(r+1)=\tfrac{1}{3}n(n+1)(n+2)$$

and

$$\sum_1^n r(r+1)(r+2)=\tfrac{1}{4}n(n+1)(n+2)(n+3).$$

(These results have some analogy to integration; the first may be compared with $\int x^2\,dx=\tfrac{1}{3}x^3$.)

Now $r^2=r(r+1)-r$, so
$$\begin{aligned} \sum_1^n r^2 &= \sum_1^n r(r+1)-\sum_1^n r \\ &= \tfrac{1}{3}n\,.\,(n+1)(n+2)-\tfrac{1}{2}n\,.\,(n+1); \end{aligned}$$

$$\therefore\ \sum_1^n r^2=\tfrac{1}{6}n\,.\,(n+1)(2n+1).$$

Extension of this method enables $\sum\limits_1^n r^3$ to be found, making use of the results for $\sum\limits_1^n r$ and $\sum\limits_1^n r^2$; similarly $\sum\limits_1^n r^4$, etc., can be found in succession, each summation depending on the results of the previous ones.

Again, to find the sum of n terms of the series

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots,$$

of which the rth term is $r(r+2)$, we see that the sum required is $\sum_1^n r(r+2)$, i.e. $\sum_1^n r^2 + 2\sum_1^n r$, and knowing $\sum_1^n r^2$ and $\sum_1^n r$ the sum can be found as $\frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1)$,

$$\text{i.e. } \tfrac{1}{6}n(n+1)(2n+7).$$

Check : When $n=1$ this result gives $\frac{1}{6}(1)(2)(9)$, i.e. 3 and $u_1 = 3$.

Use of Undetermined Coefficients

It may have been noticed in the previous work that if $f(n)$ is an integral function of n of degree k, then the degree of $f(n) - f(n-1)$ is $k-1$.

For example,

$$n^3 - (n-1)^3 \text{ is } 3n^2 - 3n + 1 \text{ of degree } 2,$$
$$n^4 - (n-1)^4 \text{ is } 4n^3 - 6n^2 + 4n - 1 \text{ of degree } 3,$$

and in general $n^k - (n-1)^k$ is kn^{k-1} + lower terms, and so is of degree $k-1$.

$\therefore$ if $S_n - S_{n-1}$ is an integral function of degree $k-1$, it is reasonable to assume that S_n is of degree k.

Therefore the series $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots$ may be summed to n terms as follows:

Since the nth term, which is $S_n - S_{n-1}$, is $n(n+2)$, it is reasonable to assume that S_n is an integral function of degree 3.

Let $S_n = An^3 + Bn^2 + Cn + D$ where A, B, C, D are, as yet, undetermined.

Then
$$\begin{aligned} S_n - S_{n-1} &= A(3n^2 - 3n + 1) + B(2n-1) + C \\ &= 3An^2 + (2B - 3A)n + A - B + C. \end{aligned}$$

If this is the same as $n^2 + 2n$ for all values of n, we have

$$3A = 1,\ 2B - 3A = 2 \text{ and } A - B + C = 0.$$

These give $A = \frac{1}{3}$, $2B = 3$ or $B = \frac{3}{2}$ and $C = B - A = \frac{7}{6}$.

$$\therefore\ S_n = \tfrac{1}{3}n^3 + \tfrac{3}{2}n^2 + \tfrac{7}{6}n + D.$$

Now $S_1 = 1 \cdot 3 = \frac{1}{3} + \frac{3}{2} + \frac{7}{6} + D$; $\quad\therefore\ D = 0$.

Hence $S_n = \dfrac{n}{6}(2n^2 + 9n + 7) = \dfrac{n}{6}(n+1)(2n+7)$ as before.

Three Available Methods

There are thus three available methods to find the sum of a series whose nth term is an integral function of n:

I. Use the sums for n, $n(n+1)$, $n(n+1)(n+2)$, etc., for each of which the sum is the next in the series divided by its number of factors.

II. Use the sum for $n, n^2, n^3, n^4, \ldots$, having memorised them. (This should be done as far as n^3 at least.)

III. Assume S_n to be an integral function of n of degree one higher than u_n with undetermined coefficients.

Example I. Find S_n if $u_r = r^3$.

Assume $\quad S_n = An^4 + Bn^3 + Cn^2 + Dn + E.$

Then $\quad S_{n-1} = A(n-1)^4 + B(n-1)^3 + C(n-1)^2 + D(n-1) + E.$

$\therefore\ u_n \equiv n^3 \equiv A(4n^3 - 6n^2 + 4n - 1) + B(3n^2 - 3n + 1) + C(2n - 1) + D.$

Equating coefficients of powers of n:

$$\begin{aligned} 4A = 1; &\quad \therefore\ A = \tfrac{1}{4}, \\ -6A + 3B = 0, &\quad \therefore\ B = \tfrac{1}{2}; \\ 4A - 3B + 2C = 0, &\quad \therefore\ C = \tfrac{1}{4}; \\ -A + B - C + D = 0, &\quad \therefore\ D = 0; \end{aligned}$$

$$\therefore\ S_n = \tfrac{1}{4}(n^4 + 2n^3 + n^2) + E.$$

Since $S_1 = 1^3 = 1$ it follows that $1 = \frac{1}{4}(1 + 2 + 1) + E$, and so $E = 0$.

$$\therefore\ S_n = \tfrac{1}{4}n^2(n+1)^2 = \{\tfrac{1}{2}n(n+1)\}^2.$$

Note that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \{1 + 2 + 3 + \ldots + n\}^2$, which is a surprising result and worth remembering.

Example II. Find $\sum\limits_1^n (2r^3 + 3r^2 - r)$.

1st solution: $\displaystyle\sum_1^n (2r^3 + 3r^2 - r) = 2\sum_1^n r^3 + 3\sum_1^n r^2 - \sum_1^n r$

$$\begin{aligned} &= 2\,\frac{n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{2}\{n(n+1) + (2n+1) - 1\} \\ &= \tfrac{1}{2}n^2(n+1)(n+3). \end{aligned}$$

2nd solution: Let $2r^3 + 3r^2 - r \equiv a(r)(r+1)(r+2) + b(r)(r+1) + c(r)$.

Equating coefficients gives $a = 2$;

$$\begin{aligned} 3a + b = 3, &\quad \therefore\ b = -3; \\ 2a + b + c = -1, &\quad \therefore\ c = -2. \end{aligned}$$

$$\therefore\ u_r = 2(r)(r+1)(r+2) - 3(r)(r+1) - 2(r).$$

$$\therefore\ S_n=\frac{2(n)(n+1)(n+2)(n+3)}{4}-\frac{3(n)(n+1)(n+2)}{3}-\frac{2(n)(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\{(n+2)(n+3)-2(n+2)-2\}$$

$$=\tfrac{1}{2}n^2(n+1)(n+3) \text{ as before.}$$

Examples 86

Use the identity $S_n-S_{n-1}=u_n$ to find u_n in the following cases, and find whether the result of putting $n=1$ in S_n and u_n gives the same result.

1. $S_n=n(n+1)$. 2. $S_n=n(n-1)$. 3. $S_n=n(2n+1)$.
4. $S_n=n(n+1)(n+2)$. 5. $S_n=n(n+1)(2n+7)$.
6. $S_n=n(n+1)(n+2)(n+3)$.
7. Show that if $S_n=[\frac{1}{2}n(n+1)]^2$, then $u_n=n^3$ and deduce that

$$\sum_1^n r^3=(\sum_1^n r)^2.$$

8. Find the sum of the first 20 terms and also of the first n terms of the series:

   (i) $1.2+2.3+3.4+\ldots$; (ii) $1.3+2.4+3.5+\ldots$;
   (iii) $2.5+4.7+6.9+\ldots$; (iv) $3.7+5.9+7.11+\ldots$;
   (v) $1.2.4+2.3.5+3.4.6+\ldots$;
   (vi) $1.2.5+2.3.6+3.4.7+\ldots$.

9. Find the following sums:

   (i) $1^2+2^2+3^2+\ldots+50^2$; (ii) $1^2+3^2+5^2+\ldots+79^2$;
   (iii) $2^2+4^2+6^2+\ldots 100^2$; (iv) $3^2+6^2+9^2+\ldots+99^2$.

10. Find the sum of the cubes of the whole numbers from 1 to 30. Deduce the following sums:

    (i) $2^3+4^3+6^3+\ldots+60^3$; (ii) $3^3+6^3+9^3+\ldots+90^3$.

11. Find (i) $\sum_1^n\{r^3+2r^2+r\}$; (ii) $\sum_1^n\{r^3-3r^2+3r-1\}$.
12. Find the sum of the squares of all the numbers less than 100 which are not multiples of either 2 or 3.
13. Show that $(r+1)(r+2)(r+3)$

    $=\frac{1}{4}[(r+1)(r+2)(r+3)(r+4)-r(r+1)(r+2)(r+3)]$.

    Use this to find the sum to n terms of the series

    $$2.3.4+3.4.5+4.5.6+\ldots.$$

14. Use the method illustrated in No. 13 to find the sum to n terms of:

    (i) $1.4.7+4.7.10+7.10.13+\ldots$;
    (ii) $2.4.6.8+4.6.8.10+6.8.10.12+\ldots$.

15. Show that $\dfrac{1}{r(r+1)}$ can be written as $\dfrac{1}{r}-\dfrac{1}{r+1}$.

Hence find the sum to n terms of the series $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\dots$.

Has this series a limiting sum? If so, what is it?

16. By expressing the rth term as the difference of two fractions, find the sum to n terms of each of the following series. Also give the limiting sum, if any.

(i) $\dfrac{1}{4.7}+\dfrac{1}{7.10}+\dfrac{1}{10.13}+\dots;$

(ii) $\dfrac{1}{3.7}+\dfrac{1}{7.11}+\dfrac{1}{11.15}+\dots;$

(iii) $\dfrac{1}{5.9}+\dfrac{1}{9.13}+\dfrac{1}{13.17}+\dots;$

(iv) $\dfrac{1}{2.3.4}+\dfrac{1}{3.4.5}+\dfrac{1}{4.5.6}+\dots.$

Series of Fractions

In Examples 86, Nos. 15 and 16, cases have been given where u_n, a fractional function of n, was expressed as a difference of two fractions in the form $f(n)-f(n-1)$, and thus $\Sigma\, u_n$ found.

Other cases will now be discussed.

Example I. Sum the series

$$\frac{1}{7.11.15}+\frac{1}{11.15.19}+\frac{1}{15.19.23}+\dots \quad \text{to } n \text{ terms,}$$

i.e. to
$$\frac{1}{(4n+3)(4n+7)(4n+11)}.$$

Using the identities

$$\frac{1}{7.11.15}=\frac{1}{8}\left(\frac{1}{7.11}-\frac{1}{11.15}\right),$$
$$\frac{1}{11.15.19}=\frac{1}{8}\left(\frac{1}{11.15}-\frac{1}{15.19}\right),$$
$$\dots\dots\dots\dots\dots\dots,$$
$$\frac{1}{(4n+3)(4n+7)(4n+11)}=\frac{1}{8}\left(\frac{1}{(4n+3)(4n+7)}-\frac{1}{(4n+7)(4n+11)}\right),$$

the required sum is seen to be

$$\frac{1}{8}\left\{\frac{1}{7.11}-\frac{1}{(4n+7)(4n+11)}\right\}.$$

Note that the 8 is the product of 4, the common difference of the A.P. and 2, the number of factors left in the denominators.

Note also that the limit of S_n as $n \to \infty$ is $\frac{1}{8} \cdot \frac{1}{7 \cdot 11}$.

This method depends on the factors of the denominator being in A.P. and the first of each set of factors forming the same A.P.

Example II. Find $\sum_{1}^{n} \frac{1}{(4r+3)(4r+7)(4r+15)}$.

Here the A.P. of the first factors in the denominator 7, 11, 15, ... is not adhered to in the 3 factors of each denominator for in the rth term above, $4r+11$ is omitted.

To use the method it must be put in above and below and the numerator transformed thus $4r+11 = 4r+15-4$:

$$\frac{4r+11}{(4r+3)(4r+7)(4r+11)(4r+15)}$$
$$= \frac{1}{(4r+3)(4r+7)(4r+11)} - \frac{4}{(4r+3)(4r+7)(4r+11)(4r+15)},$$

whence it can be shown, using $\frac{4}{12} = \frac{1}{3}$, that

$$S_n = \frac{1}{8} \cdot \frac{1}{7} \cdot \frac{1}{11} - \frac{1}{3} \cdot \frac{1}{7} \cdot \frac{1}{11} \cdot \frac{1}{15} - \frac{1}{8} \frac{1}{(4n+7)(4n+11)}$$
$$+ \frac{1}{3} \frac{1}{(4n+7)(4n+11)(4n+15)}.$$

Here the first two terms give the limit of S_n as $n \to \infty$.

Examples 87

Find the sum of the following series:

1. $\sum_{1}^{n} \frac{1}{r(r+1)(r+2)}$.

2. $\sum_{1}^{n} \frac{1}{r(r+2)}$.

3. $\sum_{1}^{n} \frac{1}{(3r+1)(3r+4)(3r+7)}$.

4. $\sum_{10}^{n} \frac{1}{(r-2)r(r+2)}$.

5. Sum to n terms the series:

$$\frac{1}{1 \cdot 2 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 6} + \dots.$$

6. Find a and b so that

$$\frac{6n+7}{(2n-1)(2n+1)(2n+3)} \equiv \frac{an+b}{(2n-1)(2n+1)} - \frac{a(n+1)+b}{(2n+1)(2n+3)},$$

and hence sum to n terms the series

$$\frac{13}{1 \cdot 3 \cdot 5} + \frac{19}{3 \cdot 5 \cdot 7} + \frac{25}{5 \cdot 7 \cdot 9} + \dots.$$

7. Show that $\qquad \frac{1}{n!} - \frac{1}{(n+1)!} = \frac{n}{(n+1)!}$.

Hence sum to n terms the series $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots$.

8. What are the limits of S_n as $n \to \infty$ for the series in Nos. 1 to 7?

9. Find the first four terms of the series for which

(i) $S_n = \dfrac{n}{2n+3}$; (ii) $S_n = \dfrac{n}{5n+2}$.

In each case also write down the nth term.

10. If S_n is taken to be $\dfrac{1}{n!}$ and $u_n = S_n - S_{n-1}$ as usual, show that the series $1 - \dfrac{1}{2!} - \dfrac{2}{3!} - \dfrac{3}{4!} - \dots$ is obtained if $u_1 = S_1$.

Sum the series of negative terms (i) to 4 terms (ii) to n terms.

[No. 10 illustrates the need for care with the first term; the series starts with u_1 and $u_1 = S_1 = 1/1!$ here. If however we put $n = 1$ in $u_n = S_n - S_{n-1}$ to give $u_1 = S_1 - S_0$, this leads to $u_1 = 1/0! - 1/1! = 0$ since 0! has already been defined as 1.]

Mathematical Induction

A theorem thought to be true for all values of the integer n can be proved true by showing that

(i) if it is true for $n = r$, then it is also true for $n = r + 1$;
(ii) it is true if $n = 1$,

for from (i) and (ii) it follows that the theorem is true for $n = 2, 3, 4, \dots$, in succession.

This process is called "mathematical induction".

Example I. Prove that the sum of the first n odd numbers is n^2.

Assume that $1 + 3 + 5 + \dots (2r - 1) = r^2$. ....(a)

Add the next odd number to both sides.

Then $$1 + 3 + 5 + \dots + (2r - 1) + (2r + 1) = r^2 + 2r + 1$$
$$= (r + 1)^2.$$

This is the same result as (a) with r replaced by $r + 1$.

Also $1 = 1^2$, $1 + 3 = 2^2$, i.e. the result is true for $n = 2$ and hence true for $n = 3$, $n = 4$, etc., in succession.

$\therefore$ the result is true for all integral values of n.

Note. It will be seen that the method is equivalent to using

$$(r + 1)^2 - r^2 = 2r + 1,$$

i.e. to differencing r^2.

Example II. Prove by induction * that

$$\sum_1^n r^2(r + 1) = \tfrac{1}{12}n(n + 1)(n + 2)(3n + 1). \quad \text{...............(b)}$$

* Induction in general language means the inferring of a general law from various particular instances, as when we infer that heat expands bodies, from the observed result that heat expands some bodies. There is in it some uncertainty which is not the case with the more limited "mathematical induction" (heat does not expand ice).

Assume the result true for $n=r$ and add the next term to each side. We must show that

$$\tfrac{1}{12}r(r+1)(r+2)(3r+1)+(r+1)^2(r+2)=\tfrac{1}{12}(r+1)(r+2)(r+3)(3r+4).$$

Now
$$\begin{aligned}&\tfrac{1}{12}(r+1)(r+2)\{(r+3)(3r+4)-r(3r+1)\}\\ &=\tfrac{1}{12}(r+1)(r+2)\{3r^2+13r+12-3r^2-r\}\\ &=\tfrac{1}{12}(r+1)(r+2)\,12(r+1)=(r+1)^2(r+2).\end{aligned}$$

This proves that we can move on from $n=r$ to $n=r+1$.

But if $n=1$, L.H.S. of (b) $=2$ and R.H.S. $=\tfrac{1}{12}\,2\cdot3\cdot4, \ldots$.

$\therefore$ the result is true for $n=1$ and hence for $n=2, 3, 4, \ldots$ in succession. The theorem is therefore proved.

Note that the method of mathematical induction is only useful for proving a *stated* result or one which might be suspected; it is of no use for discovering the result to be proved.

Examples 88

Use the method of mathematical induction to prove the following results (Nos. 1 to 5):

1. (i) $\sum_1^n r^2=\tfrac{1}{6}n(n+1)(2n+1)$;

   (ii) $1^2+3^2+5^2+\ldots$ to n terms $=\tfrac{1}{3}n(4n^2-1)$.

2. $\sum_{r=1}^{n}\dfrac{12r-5}{(2r-1)(2r+1)(2r+3)}=1\tfrac{11}{12}-\dfrac{24(n+1)-11}{4(2n+1)(2n+3)}$.

3. Amount at compound interest on £P for n years at $r\% =P\left(1+\dfrac{r}{100}\right)^n$.

4. n^3-n is divisible by 6.

5. $7^{2n}-48n-1$ is divisible by 48^2 ($=2304$).

6. For the series

$$1+p+\frac{p(p+1)}{2!}+\frac{p(p+1)(p+2)}{3!}+\ldots$$

   write down the values of S_2, S_3, S_4 and guess a value for S_r. Prove the result by induction.

7. Prove by induction that the sum of n terms of the series whose rth term is $r(r+2)(r+4)$ is $\tfrac{1}{4}n(n+1)(n+4)(n+5)$.

8. Prove by induction that x^n-a^n has $(x-a)$ as a factor for integral values of n.

   Verify that the same method fails to show that x^n+a^n has $(x+a)$ as a factor.

   [Use $x^{n+1}\mp a^{n+1}=(x^n\mp a^n)(x+a)-ax(x^{n-1}\mp a^{n-1})$. This shows that a factor for two consecutive values of n will be a factor always.]

Power Series

If $S_n \equiv a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$ where a_1, a_2, etc., are constants independent of x the series is called a *power series*.

If a_n is an integral function of n, a formula for S_n can be found by using the same method by which the sum was found for the G.P. $x + x^2 + \dots + x^n$, which in essence consisted in multiplying S_n by $1 - x$.

The process will be seen by an example.

Find $\sum\limits_1^n (r^2 - 3r + 3)x^r$.

For the series $S_n = x + x^2 + 3x^3 + 7x^4 + \dots + (n^2 - 3n + 3)x^n$;

$$\therefore\ xS_n = \quad x^2 + x^3 + 3x^4 + \dots + \{(n-1)^2 - 3(n-1) + 3\}x^n + (n^2 - 3n + 3)x^{n+1}.$$

$$\therefore\ S_n(1 - x) = x + 0x^2 + 2x^3 + 4x^4 + \dots + (2n-4)x^n - (n^2 - 3n + 3)x^{n+1};$$

$$\therefore\ S_n(1 - x) - x + (n^2 - 3n + 3)x^{n+1} = \sum_2^n (2r - 4)x^r. \quad \dots\dots\dots\text{(i)}$$

Again, let $S' = 0x^2 + 2x^3 + 4x^4 + \dots + (2n-4)x^n$,

then $\quad xS' = \quad 0x^3 + 2x^4 + \dots + \{2(n-1) - 4\}x^n + (2n-4)x^{n+1}$,

$$\therefore\ (1 - x)S' = 2x^3 + 2x^4 + \dots + 2x^n - (2n-4)x^{n+1}.$$

$$\therefore\ (1 - x)S' + (2n-4)x^{n+1} = 2x^3(1 + x + \dots + x^{n-3}),$$

i.e. $\quad (1 - x)S' + (2n-4)x^{n+1} = 2x^3 \cdot \dfrac{1 - x^{n-2}}{1 - x} \quad \dots\dots\dots\dots\dots\text{(ii)}$

From (i) and (ii) S_n is quickly found.

The result is

$$S_n = \frac{x - (n^2 - 3n + 3)x^{n+1}}{1 - x} - \frac{(2n-4)x^{n+1}}{(1 - x)^2} + \frac{2(x^3 - x^{n+1})}{(1 - x)^3}.$$

In general it will be seen that multiplication by $1 - x$ gives a result consisting of a term at each end of the original series and a new series for which the coefficient of x^r is one degree lower in r than at first.

Repetition of this process will eventually lead to a set of isolated terms and a series of terms in G.P. which can be summed.

Examples 89

Sum to n terms each of the following series (Nos. 1 to 8):

1. $1 + 2x + 3x^2 + 4x^3 + \dots + (r+1)x^r + \dots$.
2. $1 + 3x + 5x^2 + 7x^3 + \dots + (2r+1)x^r + \dots$.
3. $1 - 2x + 3x^2 - 4x^3 + \dots$.
4. $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \dots$.

5. $1-\frac{4}{5}+\frac{7}{5^2}-\frac{10}{5^3}+\dots$.

6. $1+2^2x+3^2x^2+4^2x^3+\dots$.

7. (i) $1-2^2x^2+3^2x^4-4^2x^6+\dots$, (ii) $1-2^3x+3^3x^2-4^3x^3+\dots$.

8. $1+(2^2-1)x+(3^2-1)x^2+(4^2-1)x^3+\dots$.

9. Find $\sum_{r=1}^{n} r(r-1)x^{r-1}$.

10. Find $\sum_{r=1}^{n}(r^2+2r-1)x^{r-1}$.

Note. No. 1 can be solved thus:

$$x+x^2+x^3+\dots+x^n=\frac{x(1-x^n)}{1-x};$$

$$\therefore \text{ series No. } 1=\frac{d}{dx}\left(\frac{x-x^{n+1}}{1-x}\right).$$

Recurrence Equations and Recurring Series

Sometimes the terms u_1, u_2, ... of a sequence are connected by an equation of the form

$$u_r+au_{r-1}+bu_{r-2}=0 \quad \text{..........................(i)}$$

where a and b are numbers and it is required to find u_n (u_1 and u_2 being given) and also to find $\sum_{1}^{n} u_r x^{r-1}$.

(i) is called a *recurrence* equation and the series $u_1+u_2x+u_3x^2\dots$ a *recurring* series.

The method used is illustrated by an example.

If $u_1=3$, $u_2=1$ and $u_r-2u_{r-1}-3u_{r-2}=0$, find u_n.

$u_3=2u_2+3u_1=11$; $u_4=2u_3+3u_2=22+3=25$, and in like manner succeeding terms are determined uniquely.

Now suppose $u_r=y^r$, and so $y^r-2y^{r-1}-3y^{r-2}=0$.

Hence $y^2-2y-3=0$, giving $y=3$ or -1, so that either 3^r or $(-1)^r$ for u_r satisfies the equation $u_r-2u_{r-1}-3u_{r-2}=0$.

Neither of these values, however, will give $u_1=3$ and $u_2=1$, but if we take $u_r=A\,.\,3^r+B(-1)^r$ we can find values of A and B to fit.

The equations for A and B are

$$\left.\begin{array}{ll}\text{when } r=1: & A\,.\,3+B(-1)=3\\ \text{when } r=2: & A\,.\,9+B(-1)^2=1\end{array}\right\}\text{ giving } A=\tfrac{1}{3},\ B=-2.$$

Thus the solution is

$$u_n=\tfrac{1}{3}(3)^n-2(-1)^n, \text{ i.e. } u_n=3^{n-1}+2(-1)^{n-1}.$$

Again $\sum_{1}^{n} u_r x^{r-1} = (1+3x+3^2x^2+\ldots \text{ to } n \text{ terms})$

$$+2(1-x+x^2 \ldots \text{ to } n \text{ terms})$$

$$= \frac{1-(3x)^n}{1-3x} + 2 \cdot \frac{1-(-x)^n}{1+x},$$

using the formula for G.P.'s.

In this example the relation (i) is between three consecutive terms and is said to be of the *second order*; the relation

$$u_r + au_{r-1} + bu_{r-2} + cu_{r-3} = 0$$

is a *third order* relation.

An alternative method of using the recurrence equation is given below, in which we multiply S_n by $(1+ax+bx^2)$, which is called the "*scale of relation*" corresponding to $u_r + au_{r-1} + bu_{r-2} = 0$.

Here the scale of relation is $1-2x-3x^2$, since the recurrence equation is $u_r - 2u_{r-1} - 3u_{r-2} = 0$.

$$\begin{aligned} S_n &= 3 + x + 11x^2 + 25x^3 \ldots + u_n x^{n-1}, \\ -2xS_n &= \quad -6x - 2x^2 - 22x^3 \ldots - 2u_{n-1}x^{n-1} - 2u_n x^n, \\ -3x^2S_n &= \qquad\quad -9x^2 - 3x^3 \ldots - 3u_{n-2}x^{n-1} - 3u_{n-1}x^n - 3u_n x^{n+1}. \end{aligned}$$

Adding these three rows we notice that where the columns are complete the sum is zero because of the given recurrence equation.

$$\therefore\ (1-2x-3x^2)S_n = 3 - 5x - (2u_n + 3u_{n-1})x^n - 3u_n x^{n+1}.$$

Hence $\quad S_n = \dfrac{3-5x}{1-2x-3x^2} - \dfrac{(2u_n+3u_{n-1})x^n + 3u_n x^{n+1}}{1-2x+3x^2}$ ....(A)

The first fraction here is called the *generating function*, and it can be used to find the form of u_r.

By partial fractions $\dfrac{3-5x}{1-2x-3x^2} \equiv \dfrac{1}{1-3x} + \dfrac{2}{1+x}$.

The rth terms in the expansions of $\dfrac{1}{1-3x}$ and $\dfrac{2}{1+x}$ are $(3x)^{r-1}$ and $2(-x)^{r-1}$, and so $u_r = 3^{r-1} + 2(-1)^{r-1}$ as before.

Sum to infinity

In the formula (A) above for S_n, the second fraction, by substituting the values found for u_n and u_{n-1} may be reduced to

$$-\frac{\{3^n + 2(-1)^n\}x^n + \{3^n + 6(-1)^{n-1}\}x^{n+1}}{1-2x-3x^2}$$

This should be checked by the student.

If x is sufficiently small, for example if $x = \frac{1}{10}$, this expression will have zero as its limit as $n \to \infty$.

In this case the sum to infinity of the series is

$$\frac{3-5x}{1-2x-3x^2}=\frac{2\cdot 5}{\cdot 77}\simeq 3\cdot 25.$$

For the series to converge to a finite sum, x need not be as small as $\frac{1}{10}$; it will be sufficient that $|3x|<1$.

The point to note is that, in cases similar to this, if x is sufficiently small, the series will converge and the generating function is also the sum to infinity.

The series $\Sigma\, n^2x^{n-1}$ is a recurring series whose scale of relation is $(1-x)^3$ since $n^2-3(n-1)^2+3(n-2)^2-(n-3)^2\equiv 0$.

Hence the series Σn^2x^{n-1} may be summed as follows:

$$\begin{aligned}
S_n &= 1+2^2x+3^2x^2 + 4^2x^3+\dots+n^2x^{n-1},\\
-3xS_n &= -3x-3\cdot 2^2x^2-3\cdot 3^2x^3-\dots-3(n-1)^2x^{n-1}-3n^2x^n,\\
+3x^2S_n &= 3x^2+3\cdot 2^2x^3+\dots+3(n-2)^2x^{n-1}\\
&\qquad +3(n-1)^2x^n+3n^2x^{n+1},\\
-x^3S^n &= -x^3-\dots-(n-3)^2x^{n-1}\\
&\qquad -(n-2)^2x^n-(n-1)^2x^{n+1}-n^2x^{n+2}\\
\therefore\ S_n(1-x)^3 &= 1+x+0x^2+0x^3+\dots+0x^{n-1}-(n+1)^2x^n\\
&\qquad +(2n^2+2n-1)x^{n+1}-n^2x^{n+2},
\end{aligned}$$

$$S_n=\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}.$$

Those who have worked No. 6 of Examples 89, proceeding step by step, will see that the above work is much easier.

As in the previous example, if x is sufficiently small, the series will have a "sum to infinity" which will be $(1+x)/(1-x)^3$.

Notice that if a_n is any second degree function of n, the series will be a recurring one with scale of relation $(1-x)^3$.

Similarly if a_n is a third degree function the scale of relation will be $(1-x)^4$. Nos. 9 and 10 of Examples 89 can therefore be worked in one stage, each using $(1-x)^3$ as the multiplier.

Examples 90.

1. Given the recurrence equation $u_n-4u_{n-1}+3u_{n-2}=0$, express u_5 in terms of u_1 and u_2.
2. If in the sequence $u_1, u_2, u_3, \dots$ it is given that $u_r=3u_{r-1}+4u_{r-2}$, show that $u_5=39u_2+64u_1$, and express u_6 in terms of u_1 and u_2.
3. Show that if $u_n=3+2^n$ then u_n satisfies the recurrence equation $u_n=3u_{n-1}-2u_{n-2}$ and write down the first four terms of the sequence.

4. Given the recurrence equation $u_n + u_{n-1} = 12u_{n-2}$ with the initial values $u_1 = 2$, $u_2 = 34$, find u_n and show that after u_2 the terms are alternately negative and positive.
5. If u_r satisfies the recurrence equation $u_r - 3u_{r-1} + 2u_{r-2}$ show that if $u_1 = 3$, $u_2 = 5$, then $u_r = 2^r + 1$, and find u_r if $u_1 = 7$ and $u_3 = 43$.
6. Sum to n terms the recurring series
$$u_1 + u_2 + \dots + u_n$$
being given that $u_r = 7u_{r-1} - 12u_{r-2}$ and that u_1 and u_2 are 1 and 2 respectively.
7. Sum to n terms the series
$$x - 3x^2 + 7x^3 - 15x^4 + \dots - (-1)^r(2^r - 1)x^r + \dots$$
being given that the scale of relation is $1 + 3x + 2x^2$, and show that if x is sufficiently small, the limiting sum is x.
8. Sum to n terms the series
$$1 + 8x + 10x^2 + 26x^3 + 46x^4 + \dots$$
being given that the scale of relation is of the form $1 + px + qx^2$.
[First find p and q from equations such as $46 + 26p + 10q = 0$.]
9. If $u_r = 2 \cdot 3^r - 3 \cdot 2^r$, what is the recurrence equation satisfied by u_r? Write down the first four terms of the series and show that the recurrence equation can be found from them.

If $S_n = \sum_1^n u_r x^{r-1}$ show that the series can be obtained from the partial fractions equivalent to u_2/(scale of relation).
10. Find the coefficient of x^r in the recurring series
$$4 - 6x + 10x^2 - 30x^3 + 34x^4 - 126x^5 + \dots,$$
being given that there is a relation of the form
$$u_r + au_{r-1} + bu_{r-2} + cu_{r-3} = 0$$
between the coefficients.

What is the generating function?

Examples 91 : Miscellaneous

1. Find the series whose sums to n terms are :
$$\text{(i) } n(n+5)\,; \quad \text{(ii) } \frac{1}{n(n+1)}\,; \quad \text{(iii) } n^3,$$
in each case writing down the first four terms and the rth term.
2. (i) Calculate the value of $1^3 + 2^3 + 3^3 + \dots + 10^3$, and also express 10^3 as the difference of two squares.

(ii) Express n^3 as the difference of two squares.
3. Express $6^3 + 7^3 + \dots + 11^3$ as the difference of two squares and explain how your method enables the sum of the cubes of any set of consecutive numbers to be expressed as the difference of two squares.

4. Express $\frac{1}{4}(n+2)^2(n+3)^2 - \frac{1}{4}(n-2)^2(n-3)^2$ as the sum of 5 cubes, and show that $45^2 - 28^2$ is the sum of two cubes.

5. Find the sum to n terms of the series whose rth terms are

$$\text{(i)}\ \frac{1}{r(r+3)}; \qquad \text{(ii)}\ \frac{1}{(2r+1)(2r+3)(2r+5)}.$$

6. Use the result $(n+1)! - n! = n \cdot n!$ to sum to n terms the series $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots$.

7. Show that $\dfrac{1}{(n-2)!} - \dfrac{1}{n!} = \dfrac{n^2-n-1}{n!}$,

and hence find the sum of the series

$$\frac{1}{2!} + \frac{11}{4!} + \frac{29}{6!} + \dots + \frac{379}{20!}.$$

8. Find the sum of the numbers from 50 to 250 inclusive which are not multiples of 3.

9. The sum of n terms of a series is $3n^2$ for $n = 1, 2, 3, \dots$. Find the rth term of the series. What is the first term?

Calculate the sum of all numbers between 0 and 201 which are multiples of 5 or 7; that is, find the sum of the series

$$5 + 7 + 10 + 14 + 15 + \dots + 35 + \dots + 200. \qquad \text{(O. \& C.)}$$

10. Prove that, if n is even, the sum of the first n terms of the series

$$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots$$

is $\frac{1}{2}n(n+1)^2$, and find the sum if n is odd. (O. & C.)

11. If $1^2 + 2^2 + 3^2 + \dots + n^2 = An(n-1)(n-2) + Bn(n-1) + Cn + D$

for all values of n, find the values of the constants A, B, C and D.

Sum the series

$$1 + 3 \cdot 2 + 5 \cdot 2^2 + 7 \cdot 2^3 + \text{to } n \text{ terms.} \qquad \text{(L.)}$$

12. Find the values of a and b in terms of n if

$$(x-n+1)^3 - (x-n)^3 \equiv 3x^2 + ax + b$$

for all values of x.

By adding together equations of this form for $x = 0$ and $n = 1, 2, \dots, r$, obtain an expression for

$$\sum_{n=1}^{r} n^2. \qquad \text{(L.)}$$

13. Verify that

$$r^2(r+1)^2 - (r-1)^2 r^2 = 4r^3,$$

and deduce, or prove otherwise, that

$$1^3 + 2^3 + \dots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Prove that

$$1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n^2-1),$$

and find the sum of the cubes of all odd numbers less than 50 that are not multiples of 5. (O. & C.)

14. The odd numbers are bracketed in groups thus:

$$(1); \quad (3, 5); \quad (7, 9, 11); \quad (13, 15, 17, 19); \ldots$$

the nth bracket containing n numbers. Find the sum of the numbers in the nth bracket. (N.)

15. Find the sum of all the odd numbers which are less than $6n+1$ and are not multiples of 3.

16. Find A, B, independent of n, such that

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n(n+1)} + \frac{B}{(n+1)(n+2)}$$

for all values of n.

Prove that the sums of the first n terms of the series whose rth terms are $\frac{1}{(r+1)(r+2)}$ and $\frac{1}{r(r+1)(r+2)}$ are

$$\frac{1}{2} - \frac{1}{n+2} \text{ and } \frac{1}{4} - \frac{1}{2(n+1)(n+2)}. \qquad \text{(O. \& C.)}$$

17. Find the sum of the first n terms of the series

$$1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \ldots.$$

Prove by induction that the sum of the first n terms of the series

$$\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \ldots$$

is

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n+1)}{2 \cdot 4 \ldots 2n} - 1. \qquad \text{(O. \& C.)}$$

18. Prove that

$$\sum_{r=1}^{n} r(r+1)(r+2) = \tfrac{1}{4}n(n+1)(n+2)(n+3),$$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Prove that $n=1$ is the *only* positive integer to satisfy the equation

$$24\sum_{r=1}^{n} r^3 = \sum_{r=n+1}^{2n} r(r+1)(r+2). \qquad \text{(O. \& C.)}$$

19. Find the numbers A and B, independent of n, which are such that

$$n^5 \equiv An^3(n+1)^3 + Bn^2(n+1)^2 - An^3(n-1)^3 - Bn^2(n-1)^2,$$

and use your result to find a and b when

$$1^5 + 2^5 + \ldots + n^5 \equiv \tfrac{1}{12}n^2(n+1)^2(an^2+bn-1). \qquad \text{(O. \& C.)}$$

20. (i) Prove that a, b, and c may be chosen so that

$$\sum_{r=0}^{n} (4r+1)^2 = an + 1 + b\sum_{r=0}^{n} r^2 + c\sum_{r=0}^{n} r(r+1).$$

(ii) Prove by induction that

$$\sum_{r=1}^{n} r(r+1)\ldots(r+k-1) = \frac{n(n+1)\ldots(n+k)}{k+1}. \qquad \text{(O. \& C.)}$$

21. Prove that $\{1 \cdot n + 2(n-1) + 3(n-2) + \ldots + (n-1)2 + n \cdot 1\}/n^3$ tends to a limit as $n \to \infty$ and find this limit.
[Hint. Prove $\{\ \} = \frac{1}{6}n(n+1)(n+2)$.] (N.)

22. Find the value of u_n given

(i) $u_n + u_{n-1} = 6u_{n-2}, \quad u_1 = 1, \quad u_2 = 47$;
(ii) $u_n - u_{n-1} = 6u_{n-2}, \quad u_1 = 1, \quad u_2 = 73$.

23. If each term of a series is the sum of the two previous terms (a "Fibonacci" Series), show that $S_n = (2u_n + u_{n-1} - u_2)$ and give the corresponding result for the power series $\Sigma u_r x^r$.

24. Find the sum of the cubes of the first n terms of the A.P., 2, 5, 8, 11, ...

25. If $u_n = 5u_{n-1} + 2$ and $u_1 = 3$, find u_n.

CHAPTER X

THE STANDARD POWER SERIES

The Binomial Theorem and Differentiation of Power Series

THE Binomial Theorem states that, if $|x|<1$, then

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots \quad \dots\text{(I)}$$

the $(r+1)$th term being

$$\frac{m(m-1)(m-2)\dots(m-r+1)}{r!}x^r.$$

This has been proved in Chapter VIII for the case when m is a positive integer, in which case there is no restriction on the value of x, since the number of terms is finite. It has also been proved true when $m=-1$ and the early terms of the series have been obtained for $m=-2$ and $m=\frac{1}{2}$; in these cases $|x|<1$ is essential.

We proceed to compare the results for $m=-1$ and $m=-2$.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots,$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots.$$

It will be seen that if we differentiate $(1+x)^{-1}$ we get $-(1+x)^{-2}$, and that if we differentiate the series for $(1+x)^{-1}$ term by term we get $-(1-2x+3x^2-4x^3+5x^4-\dots)$.

In this case therefore it appears that to differentiate the series term by term gives a series equal to the derivative of $(1+x)^{-1}$, the sum of the original series.

Differentiating the series for $(1+x)^{-2}$ gives another example of this, since it will be found that the series for $(1+x)^{-3}$ is obtained.

These series consist of terms in ascending powers of the variable x with constants as coefficients of these powers of x.

Such series are called "*power series*".

Let us therefore make the assumption that *if the power series concerned are convergent, it is legitimate to differentiate term by term*; so that, if

$$\left.\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots, \\ \text{then} \quad f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots. \end{aligned}\right\} \quad \dots\dots\text{(II)}$$

With this assumption it is possible to obtain the general binomial expansion without any reference to the special case of a positive integer as exponent.

Two methods of using this assumption will now be given.

First Method

If $y=(1+x)^m$ then $\dfrac{dy}{dx}=m(1+x)^{m-1}$, so that

$$(1+x)\frac{dy}{dx}=my. \qquad \text{(III)}$$

Hence, if in (II) we put $f(x)=(1+x)^m$, equation (III) gives

$$(1+x)(a_1+2a_2x+3a_3x^2+\ldots)=m(a_0+a_1x+a_2x^2+\ldots).$$

Equating coefficients :

$$a_1=ma_0,$$

$$a_1+2a_2=ma_1 \quad \text{or} \quad a_2=\frac{m-1}{2}\,a_1,$$

$$2a_2+3a_3=ma_2 \quad \text{or} \quad a_3=\frac{m-2}{3}\,a_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(r-1)a_{r-1}+ra_r=ma_{r-1} \quad \text{or} \quad a_r=\frac{m-r+1}{r}\,a_{r-1}.$$

This gives the series for $(1+x)^m$ multiplied by the constant a_0, viz.

$$(1+x)^m=a_0\left(1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\ldots\right).$$

In this, putting $x=0$ shows that $a_0=1$.

Thus the expansion stated in (I) has been obtained.

It is easy to show that, except for a constant multiplier, the solution of the (differential) equation (III) is unique.

$(1+x)^m$ is one solution ; suppose y_1 to be another, so that

$$(1+x)\frac{dy_1}{dx}=my_1.$$

Then provided $|x|<1$, so that $(1+x)^m$ cannot be zero, we can differentiate $\dfrac{y_1}{(1+x)^m}$, obtaining $\dfrac{(1+x)^m\dfrac{dy_1}{dx}-y_1\,.\,m(1+x)^{m-1}}{(1+x)^{2m}}$.

This equals
$$\frac{(1+x)\dfrac{dy_1}{dx}-my_1}{(1+x)^{m+1}}=0,$$

so that $\dfrac{y_1}{(1+x)^m}$ is a constant.

Note. The proof that, under certain conditions, a power series may be differentiated term by term, as assumed in (II), is beyond the scope of this book.

Those readers who require the proof are referred to Whittaker and Watson's *Modern Analysis*, p. 31.

Second Method

We now give an alternative method, which may be found easier, of arriving at the binomial series. Assume, as before, that with $|x|<1$,

$$(1+x)^m \equiv a_0+a_1x+a_2x^2+\dots+a_rx^r+\dots,$$

and differentiate both sides r times; this gives

$$m(1+x)^{m-1}\equiv a_1+2a_2x+3a_3x^2+\dots+r\,.\,a_rx^{r-1}+\dots,$$
$$m(m-1)(1+x)^{m-2}\equiv 2a_2+3\,.\,2a_3x+4\,.\,3\,.\,a_4x^2+\dots+r(r-1)a_rx^{r-2}+\dots$$
$$m(m-1)(m-2)(1+x)^{m-3}\equiv 3!\,a_3+4\,.\,3\,.\,2a_4x+5\,.\,4\,.\,3a_5x^2+\dots+r(r-1)(r-2)a_rx^{r-3}\dots,$$

....

$$m(m-1)(m-2)\dots(m-r+1)(1+x)^{m-r}\equiv r(r-1)(r-2)\dots 3\,.\,2\,.\,a^r+(r+1)r\dots 3\,.\,a_{r+1}x+\dots.$$

In these identities put $x=0$; the results are

$$1=a_0.$$
$$m=a_1,$$
$$m(m-1)=2a_2,$$
$$m(m-1)(m-2)\dots=3!\,a_3$$

....

$$m(m-1)(m-2)\dots(m-r+1)=r!\,a_r.$$

....

These equations give the series

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\dots+\frac{m(m-1)\dots(m-r+1)}{r!}x^r+\dots,$$

which is the Binomial Theorem.

It will be seen that if m is a positive integer the process terminates at the stage $m(m-1)(m-2)\dots(m-m+1)=m!\,a_m$, giving $a_m=1$ and showing that all subsequent coefficients are zero.

Convergence of $(1-x)^{-m}$

If x is positive and less than one and m is positive, $(1+x)^m<2^m$, and so only becomes large if m is large; but the case of $(1-x)^{-m}$ is different.

If x is nearly one, $(1-x)^{-m}$ can be very large even if m is less than (say) 10.

The series can, however, be seen to converge in any special case by the method shown in the following example.

The simplified form of the coefficients for $(1-x)^{-m}$ should be noticed.

Example I. Write down and simplify the first four terms in $(1-x)^{-m}$. If $x=\cdot 9$ and $m=7$ show that the terms do not begin to diminish till after the 55th term.

Show also that starting at the 55th term, the terms are less than those of a convergent G.P., so that the series is convergent.

Solution. $(1-x)^{-m}$

$$=1+\frac{(-m)}{1}(-x)+\frac{(-m)(-m-1)}{1\,.\,2}(-x)^2$$
$$+\frac{(-m)(-m-1)(-m-2)}{1\,.\,2\,.\,3}(-x)^3+\dots$$
$$=1+\frac{m}{1}x+\frac{m(m+1)}{1\,.\,2}x^2+\frac{m(m+1)(m+2)}{1\,.\,2\,.\,3}x^3\dots.$$

If $x=\cdot 9$ and $m=7$ these read

$$1+\frac{7}{1}\cdot\frac{9}{10}+\frac{7\,.\,8}{1\,.\,2}\left(\frac{9}{10}\right)^2+\frac{7\,.\,8\,.\,9}{1\,.\,2\,.\,3}\cdot\left(\frac{9}{10}\right)^3+\dots,$$

the $(n+1)$th term, u_{n+1}, being $\dfrac{7\,.\,8\,.\,9\dots(n+6)}{1\,.\,2\,.\,3\dots n}\cdot\left(\dfrac{9}{10}\right)^n$.

$$\therefore\ \frac{u_{n+1}}{u_n}=\frac{(n+6)\,.\,9}{n\,.\,10}=\frac{9n+54}{10n}.$$

Now $9n+54=10n$ if $n=54$, so the 54th and 55th terms are equal, being

$$\frac{7\,.\,8\,.\,9\dots 59}{1\,.\,2\,.\,3\dots 53}\cdot\left(\frac{9}{10}\right)^{53}\quad\text{and}\quad\frac{7\,.\,8\,.\,9\dots 60}{1\,.\,2\,.\,3\dots 54}\cdot\left(\frac{9}{10}\right)^{54}.$$

The multipliers to produce the following terms will be

$$\frac{61}{55}\times\frac{9}{10},\quad\frac{62}{56}\times\frac{9}{10},\quad\frac{63}{57}\times\frac{9}{10},\ \dots.$$

These steadily diminish since $\frac{n+6}{n} > \frac{(n+1)+6}{n+1}$,

i.e. since $n^2+7n+6 > n^2+7n$.

So if we take $r = \frac{61}{55} \times \frac{9}{10} = \frac{549}{550}$,

$$u_{56} = u_{55} \times r, \quad u_{57} < u_{55} \times r^2, \quad u_{58} < u_{55} \times r^3,$$

and so on.

Hence starting at the 55th term, the terms of the series are less than $u_{55}\{1+r+r^2+r^3+\dots\}$, which is a convergent G.P. since $|r|<1$.

Thus the given series is convergent ; * its value is

$$(1-\cdot 9)^{-7} = \left(\frac{1}{10}\right)^{-7} = 10^7.$$

Generalisation

$$(1-x)^{-m} = 1 + \frac{m}{1}x + \frac{m(m+1)}{1\,.\,2}x^2 + \frac{m(m+1)(m+2)}{1\,.\,2\,.\,3}x^3 + \dots,$$

the $(n+1)$th term, u_{n+1} being $\frac{m(m+1)(m+2)\dots(m+n-1)}{1\,.\,2\,.\,3\dots n}x^n$,

so that $$\frac{u_{n+1}}{u_n} = \frac{m+n-1}{n}\,.\,x.$$

This is greater than $x\ (0<x<1)$ if $m>1$; but if n is large enough for any given m, it can be shown to be less than a constant, which is itself less than one though larger than x, for instance, $\frac{1}{2}(1+x)$.

For $\frac{m+n-1}{n}\,.\,x < \frac{1+x}{2}$ if $2(m+n-1)x < n(1+x)$,

i.e. if $2(m-1)x < n(1-x)$,

which is true if $n > \frac{2(m-1)x}{1-x}$.

If p is the integer next greater than $\frac{2(m-1)x}{1-x}$, then the sum of the terms of the series from the pth term onwards will be less than $u_p(1+r+r^2+r^3+\dots)$ where $r=\frac{1}{2}(1+x)$, and so the series will be convergent.

In the case given in Example I, $(1-x)^{-7}$ with $x=\frac{9}{10}$, $\frac{x}{1-x}=9$,

and we have $n > 2\times 6\times 9$ and $r = \frac{19}{20}$.

* This argument uses a special case of the general result that a series of positive terms if "bounded above" is convergent. A diagram to illustrate the general case is given in Chapter XII.

Thus $p=109$, and the above has shown that the sum from the 109th term onwards is less than $u_{109}/(1-\frac{19}{20})$ or $20 \cdot u_{109}$.

This proves the series convergent, although we have not started from the term at which the terms begin to decrease.

Examples 92

1. By differentiating the binomial series for $(1+x)^{-2}$ obtain that for $(1+x)^{-3}$.
2. Differentiate the series for $(1-x)^{-1}$ and state the result obtained.
3. Write down the binomial series for $(1+x)^{\frac{1}{2}}$ by putting $m=\frac{1}{2}$ in (I), p. 227. By differentiating obtain that for $(1+x)^{-\frac{1}{2}}$, and verify that putting $m=-\frac{1}{2}$ in (I) gives the same result.
4. Use the binomial series to show that if $|x|<1$ then

$$\text{(i)}\ \frac{1+x^2}{(1-x)^3}=1+3x+7x^2+13x^3+\dots;$$

$$\text{(ii)}\ \frac{1+x}{(1-x)^3}=1+x+3x^2+3x^3+\dots.$$

5. Show that when the series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

is differentiated term by term the same series is obtained.

6. Assume that $(1+x)^6=a_0+a_1x+a_2x^2+\dots=y$, and find the expansion from the equation $(1+x)\dfrac{dy}{dx}=6y$, showing that it terminates at the term in x^6.
7. Show that the equation (II) can be regarded as giving the result of passing from the second series to the first by integration.

   Obtain a result by integrating both sides of

$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{5}{128}x^4+\dots.$$

8. If $(1+x)^n=c_0+c_1x+c_2x^2+\dots+c_nx^n$ where n is a positive integer, show that $c_0+c_1+c_2+\dots+c_n=2^n$.

   By differentiation and integration, find the values of

   (i) $c_1+2c_2+3c_3+\dots+nc_n$;

   (ii) $\dfrac{c_0}{1}+\dfrac{c_1}{2}+\dfrac{c_2}{3}+\dots+\dfrac{c_n}{n+1}$.

9. In the case of the series for $(1-x)^{-7}$ with $x=\cdot 9$, considered on p. 230, show that $\dfrac{u_{n+1}}{u_n}\leqslant\dfrac{99}{100}$ if $n\geqslant 60$.

10. Write down the first five terms of $(1-x)^{-6}$ and simplify them.

If $x=\frac{4}{5}$ show that the 20th and 21st terms are equal, and that starting at the 21st term, the terms of the series are less than those of a G.P. whose common ratio is $104/105$.

11. Write down the first five terms of the series for $(1-x)^{-\frac{9}{2}}$ and show that the fifth term $=\dfrac{9 \cdot 11 \cdot 13 \cdot 15}{2 \cdot 4 \cdot 6 \cdot 8}x^4$.

When do the terms begin to diminish if $x=\cdot 7$?

12. If $y=1-\dfrac{1}{x}$, expand x^n in powers of y as far as the term in y^3.

13. Write down and simplify the $(r+1)$th term of $(1+x)^{-3}$.

If $x=\cdot 9$ show that this term is numerically less than the previous one if $r>18$.

14. If $0<x<1$ and $r-1<m<r$ show that the expansion of $(1+x)^m$ consists of $r+1$ positive terms followed by negative ones.

15. If p and q are positive integers with $q>p$ and if $0<qx<1$, expand $(1-qx)^{-\frac{p}{q}}$, giving the first four terms and the general term.

Show that the ratio $(n+1)$th term/nth term $<qx$.

16. Expand $(1+x+x^2)^{\frac{1}{2}}$ in ascending powers of x to four terms:

(i) as $\{1+(x+x^2)\}^{\frac{1}{2}}$ assuming $|x+x^2|<1$ and

(ii) as $(1-x^3)^{\frac{1}{2}}/(1-x)^{-\frac{1}{2}}$.

Further Examples on the use of the Binomial Theorem

Example I. If $\dfrac{2-7x}{1-7x+12x^2}$ is expanded in powers of x, find the coefficient of x^n and the range of values of x possible.

The fraction $\equiv \dfrac{1}{1-3x}+\dfrac{1}{1-4x}$. Each of these fractions can be expanded by the Binomial Theorem, or regarded as the limiting sums of geometric series.

The first fraction is $1+3x+3^2x^2+3^3x^3+\ldots+3^nx^n+\ldots$ and is valid if $|3x|<1$,

while the second is $1+4x+4^2x^2+4^3x^3+\ldots+4^nx^n+\ldots$ and is valid if $|4x|<1$.

Consequently the given fraction is $2+(3+4)x+(3^2+4^2)x^2+\ldots$ and valid if $|x|<\frac{1}{4}$.

The coefficient of x^n is 3^n+4^n.

Example II. Expand $\sqrt{\left(\frac{1-x}{1+x}\right)}$ as far as x^5, given $|x|<1$.

The expression $=(1-x)(1-x^2)^{-\frac{1}{2}}$

$$=(1-x)\left\{1+(-\tfrac{1}{2})(-x^2)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x^2)^2+\dots\right\}$$
$$=(1-x)(1+\tfrac{1}{2}x^2+\tfrac{3}{8}x^4+\dots)$$
$$=1-x+\tfrac{1}{2}x^2-\tfrac{1}{2}x^3+\tfrac{3}{8}x^4-\tfrac{3}{8}x^5+\dots .$$

Example III. If $|x|<\frac{1}{2}$, find an expression in powers of x as far as the term in x^4 for $\dfrac{1}{(1-x)(1+2x)^2}$.

Suppose the given fraction $\equiv \dfrac{a}{1-x}+\dfrac{b}{1+2x}+\dfrac{c}{(1+2x)^2}$.

Then $1\equiv a(1+4x+4x^2)+b(1+x-2x^2)+c(1-x)$ and equating terms gives

$$\left.\begin{aligned} a+b+c&=1\\ 4a+b-c&=0\\ 4a-2b\quad&=0\end{aligned}\right\}$$

Adding these equations gives $9a=1$.

$\therefore\ a=\frac{1}{9},\ b=\frac{2}{9},\ c=\frac{6}{9}$.

[The "cover-up" rule also gives $a=\frac{1}{9}$.]

The given fraction is $\dfrac{1}{9(1-x)}+\dfrac{2}{9(1+2x)}+\dfrac{6}{9(1+2x)^2}$.

Expanding by the Binomial Theorem, $|2x|$ being <1, the fraction is the sum of the series

$$\tfrac{1}{9}\{1+x+x^2+x^3+x^4+\dots\}$$
$$+\tfrac{2}{9}\{1-2x+2^2x^2-2^3x^3+2^4x^4+\dots\}$$
$$+\tfrac{6}{9}\{1-2\,.\,2x+3\,.\,2^2x^2-4\,.\,2^3x^3+5\,.\,2^4x^4\ \dots\}$$

and so is $\frac{1}{9}\{9-27x+81x^2-207x^3+513x^4-\dots\}$

$$=1-3x+9x^2-23x^3+57x^4-\dots .$$

Recurring Series

It is worthwhile to examine Example III from another point of view.

Suppose $\dfrac{1}{(1-x)(1+2x)^2}=a_0+a_1x+a_2x^2+a_3x^3+\dots .$

Then $\quad 1\equiv(1+3x-4x^3)(a_0+a_1x+a_2x^2+a_3x^3+\dots).$

Equating coefficients gives $a_0=1$, $a_1+3a_0=0$, $a_2+3a_1=0$, $a_3+3a_2-4a_0$, etc.

$$\therefore\ a_0=1,\quad a_1=3,\quad a_2=9,\quad a_3=-23.$$

This method is quicker than that using partial fractions.

The general relation between the coefficients is

$$a_n + 3a_{n-1} - 4a_{n-3} = 0,$$

and the series obtained is a recurring series with scale of relation $1 + 3x - 4x^3$. (See p. 221.)

Example IV. Expand $(2+x)^{\frac{1}{3}}$ in ascending powers of x as far as x^3 given $|x|<2$. Put $x=\frac{1}{4}$ and hence find the value of $9^{\frac{1}{3}}$.

$$(2+x)^{\frac{1}{3}} = 2^{\frac{1}{3}}\left(1+\frac{x}{2}\right)^{\frac{1}{3}}$$

$$=2^{\frac{1}{3}}\left\{1+\frac{1}{3}\cdot\frac{x}{2}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\frac{x^2}{4}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\cdot\frac{x^3}{8}+\dots\right\}$$

$$=2^{\frac{1}{3}}\left\{1+\frac{x}{6}-\frac{x^2}{36}+\frac{5x^3}{8\cdot 3^4}-\dots\right\}.$$

Putting $x=\frac{1}{4}$,

$$\left(\frac{9}{4}\right)^{\frac{1}{3}} = 2^{\frac{1}{3}}\left\{1+\frac{1}{24}-\frac{1}{36\cdot 16}+\frac{5}{8\cdot 3^4\cdot 4^3}-\dots\right\}$$

$$9^{\frac{1}{3}} \simeq 2\left\{1+\frac{1}{24}-\frac{1}{36\cdot 16}+\frac{5}{9^2\cdot 8^3}+\dots\right\}$$

$$\begin{array}{rl} \simeq & 2\cdot 083333 \\ - & \cdot 003472 \\ + & \cdot 00024 \\ \hline \simeq & 2\cdot 08010 \end{array}$$

$\therefore\ 9^{\frac{1}{3}} = 2\cdot 0801$ to 4 dec. places.

$$\begin{array}{r} 3\ )\ \cdot 083333 \\ 8\ )\ \cdot 027777 \\ \hline \cdot 003472 \\ 5 \\ \hline 9\ )\ \cdot 017360 \\ 8\ )\ \cdot 00193 \\ \hline \cdot 00024 \end{array}$$

Examples 93

Express the following functions in partial fractions; find the first three terms and the coefficient of x^r in the power series equal to them, stating in each case the range of values for x for which the series are valid.

1. $\dfrac{3}{1+x-2x^2}$. 2. $\dfrac{3+x}{1-x^2}$. 3. $\dfrac{14x-1}{1+2x-8x^2}$.

4. $\dfrac{x+1}{6-13x+6x^2}$. 5. $\dfrac{1+2x-x^2}{1-x^2}$. 6. $\dfrac{a+b+(a-2b)x}{1-x-2x^2}$.

7. Calculate $\sqrt[3]{28}$ to 4 decimal places from $3(1+1/3^3)^{\frac{1}{3}}$.

8. Expand $(1-x^2)^{-\frac{1}{2}}$ to 3 terms and by putting $x=\frac{1}{9}$ calculate $1/\sqrt{80}$.

9. Identify the binomial series $1+2x+3x^2+4x^3+\dots$.
For what value of x is its sum equal to 10000?

10. If $|x|<1$ show that $\sqrt{\left(\dfrac{1+x}{1-x}\right)}+\sqrt{\left(\dfrac{1-x}{1+x}\right)}=2\left(1+\dfrac{1}{2}x^2+\dfrac{3}{8}x^4+\dots\right)$.

11. If $\dfrac{3 - 12x + 11x^2}{(1 - x)(1 - 2x)(1 - 3x)}$ is expanded in powers of x show that the coefficient of x^3 is 36, and that this term must numerically be less than 4/3 for the expansion to be valid.

12. Find the expansions as far as x^2 for Nos. 1 and 3 without using partial fractions. State in each case the recurrence equation between 3 consecutive coefficients.

13. If $\log y = n \log \dfrac{1+x}{1-x}$, show that $(1 - x^2)\dfrac{dy}{dx} = 2ny$. Hence if

$$\left(\frac{1+x}{1-x}\right)^n = \Sigma\, a_r x^r,$$

prove that $(r+1)a_{r+1} - 2na_r - (r-1)a_{r-1} = 0$ given that $|x| < 1$.

Obtain a corresponding relation if $\left(\dfrac{1-x}{1+x}\right)^n = \Sigma\, b_r x^r$.

14. If $\dfrac{3 - 4x}{2 - 7x + 6x^2} = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$, the series being convergent for sufficiently small values of x, prove that

$$2a_n - 7a_{n-1} + 6a_{n-2} = 0$$

for all integral values of n from 2 upwards.

Using partial fractions find the value of a_n and determine for what range of values of x the series is convergent. (B.)

15. Use the Binomial Theorem to expand both the expressions $\sqrt{(1 - 2x)}$ and $\dfrac{2 - 3x}{2 - x}$ in ascending powers of x as far as the term in x^4 in each case. State the range of values of x for which each of the expansions is valid.

Prove also, by putting $x = \frac{1}{50}$ in your expansions, that $\sqrt{6}$ differs from $\frac{485}{198}$ by less than 0·000,01. (O. & C.)

The Exponential Series

It has been explained in Chapter IV that the derivative of a^x is of the form ka^x where k is a number, depending on the value of a but not on the value of x.

Also that there is a number e (approximately 2·7) such that, in the case when $a = e$, the number $k = 1$.

$$\therefore \text{ if } y = e^x, \text{ then } \frac{dy}{dx} = e^x,$$

and e^x satisfies the equation $\dfrac{dy}{dx} = y$. ....(I)

As for the Binomial Theorem, which involved the more difficult relation $(1+x)\dfrac{dy}{dx}=my$, it is easy to show that the solution of (I) is unique except for a possible constant multiplier.

For if y_1 is a solution different from e^x, so that $\dfrac{dy_1}{dx}=y_1$, and since e^x is never zero, we can differentiate $\dfrac{y_1}{e^x}$ to give

$$\frac{d}{dx}\left(\frac{y_1}{e^x}\right)=\frac{e^x\dfrac{dy_1}{dx}-y_1e^x}{e^{2x}}=\frac{\dfrac{dy_1}{dx}-y_1}{e^x}=0,$$

so that y_1/e^x is a constant.

$$\therefore\ y_1=ke^x \text{ where } k \text{ is a constant.}$$

Now if the power series $a_0+a_1x+a_2x^2+\dots$ is a solution of (I) we have

$$a_1+2a_2x+3a_3x^2+4a_4x^3+\dots=a_0+a_1x+a_2x^2+a_3x^3+\dots,$$

and get in succession :

$$\begin{array}{ll} a_1=a_0 & \\ 2a_2=a_1 & a_2=\dfrac{1}{2}a_0 \\ 3a_3=a_2 & a_3=\dfrac{1}{2\cdot 3}a_0 \\ 4a_4=a_3 & a_4=\dfrac{1}{2\cdot 3\cdot 4}a_0, \end{array}$$

so that the solution is

$$a_0\left(1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right).$$

It follows that

$$ke^x=a_0\left(1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right).$$

Putting $x=0$ on both sides we see that $k=a_0$;

$$\therefore\ e^x=1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^r}{r!}+\dots. \qquad \text{..............(II)}$$

The series on the right is the *exponential series* and will be denoted by exp x, *this notation meaning the series.*

Thus (II), which is the Exponential Theorem, may be restated as

$$e^x = \exp x. \quad \text{.............................(II)}'$$

Note that $e = \exp 1 = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{r!} + \dots$.

Second Method

On p. 229 we gave a second method for obtaining the Binomial Series, the coefficients being found in turn by repeated differentiation followed by putting $x=0$ in the resulting identities. This method is equally effective for the Exponential Series, and its application is left as an exercise for the student; it will be discovered, however, that it is no more simple than the preceding method.

Convergence of the Exponential Series

Unlike the Binomial Series, the Exponential Series is *convergent for all values of x.*

Consider, for example, the case when $x=6$.

The first six terms are

$$1 + \frac{6}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!} + \frac{6^5}{5!}.$$

So far the terms increase steadily, the last one being $324/5 \simeq 65$.

The seventh term is equal to the sixth.

The next six terms get steadily smaller, for moving on we multiply by $\frac{6}{7}, \frac{6}{8}, \frac{6}{9}, \frac{6}{10}, \frac{6}{11}, \frac{6}{12}$, and so reach the 13th term. After this we multiply by $\frac{6}{13}, \frac{6}{14}, \frac{6}{15}$, and so on, and each term is less than half the preceding one.

Calling the 13th term u_{13}, the sum of n terms starting at the 13th is less than $u_{13}(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ to n terms), i.e. is less than

$$u_{13} \times \left\{ \frac{1}{1 - \frac{1}{2}} - \frac{(\frac{1}{2})^n}{1 - \frac{1}{2}} \right\}.$$

However far we go the sum is less than $u_{13} \times \frac{1}{1 - \frac{1}{2}}$ or $2u_{13}$, and the whole series less than sum of 1st twelve terms + twice 13th term, and so is convergent.

An exactly similar argument applies for any positive value of x.

Suppose the integer $r \geqslant x$, then if we go beyond the $2r$th term each term is less than half the previous one, so that however far we go the sum is less than

the sum of the first $2r$ terms + twice the $(2r+1)$th term.

Now consider the case when x is negative, for example -6.

Every second term is negative ; the 13th term is the same as before, and the odd-numbered terms starting there are less than

$$u_{13}(1+\tfrac{1}{4}+\tfrac{1}{16}+\ldots),$$

while the even-numbered ones are negative but are numerically less than $u_{13}(\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\ldots)$, both of which series have finite sums however far they are taken.

Thus (the first series) − (the second series) gives a finite result.

The general case of $\exp(-x)$ may be dealt with similarly.

It should be noticed that since $\exp k$ is a convergent series, the remainder after n terms can be made as small as may be wished by increasing n.

Now in the series $\exp(-k)$ the remainder consists of the same terms as for $\exp k$ but has every second one minus, so that the magnitude of the remainder at any stage in the series $\exp(-k)$ is less than that in the series $\exp k$, so that $\exp(-k)$ is the more rapidly convergent of the two.

There is no possibility of oscillation as in the case of the series $1+(-1)+(-1)^2+(-1)^3+\ldots$ since, after a finite number of terms, the terms of any exponential series diminish indefinitely.

The student must make sure of remembering that

The binomial series for $(1+x)^m$, where m is not a positive integer, is only convergent if $|x|<1$, but the exponential series exp x is convergent for all values of x.

Of the other series to be considered in this chapter, some will be like the binomial series and some like the exponential series as regards convergence.

e is not a Rational Number

For if e were a rational number $\dfrac{p}{q}$ where p and q are integers, then $q \cdot e$ would be an integer and so would $e(q!)$.

But $e \cdot q! = \text{integer} + \dfrac{1}{q+1} + \dfrac{1}{(q+1)(q+2)} + \dfrac{1}{(q+1)(q+2)(q+3)} + \ldots.$

Now

$$\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\frac{1}{(q+1)(q+2)(q+3)}+\ldots$$

is less than $\frac{1}{q+1}+\frac{1}{(q+1)^2}+\frac{1}{(q+1)^3}+\ldots$, which is a G.P. whose sum $\to \frac{1}{q}$.

$\therefore\ e \, . \, q! =$ integer + a fraction, and so the hypothesis that e is rational is false.

Example I. Calculate the value of $e^{\frac{1}{3}}$ from the series to 5 places of decimals.

The method is to get each term from the previous one by dividing, expressing the result as a decimal, and working to 7 decimal places.

$1 =$	$1{\cdot}000{,}000{,}0 \ldots$	
$\frac{1}{3} =$	$\cdot 333{,}333{,}3 \ldots$	
$\left(\frac{1}{3}\right)^2 \cdot \frac{1}{2!} =$	$\cdot 055{,}555{,}5 \ldots$	dividing by 6
$\left(\frac{1}{3}\right)^3 \cdot \frac{1}{3!} =$	$\cdot 006{,}172{,}8 \ldots$	dividing by 9
$\left(\frac{1}{3}\right)^4 \cdot \frac{1}{4!} =$	$\cdot 000{,}514{,}4 \ldots$	dividing by 12
$\left(\frac{1}{3}\right)^5 \cdot \frac{1}{5!} =$	$\cdot 000{,}034{,}3 \ldots$	dividing by 15
$\left(\frac{1}{3}\right)^6 \cdot \frac{1}{6!} =$	$\cdot 000{,}001{,}9 \ldots$	dividing by 18
$\left(\frac{1}{3}\right)^7 \cdot \frac{1}{7!} =$	$\cdot 000{,}000{,}1 \ldots$	dividing by 21
	$1{\cdot}395{,}6123$	

It is unnecessary to go further, as the next term starts with eight zeros.

$$e^{\frac{1}{3}} = 1{\cdot}395{,}61 \text{ to 5 decimal places.}$$

Example II. Find the first five terms in the expansion of $\frac{1-x}{e^x}$ in powers of x, and state the rth term.

$$\frac{1-x}{e^x} = (1-x)e^{-x} = (1-x)\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

$$- x + x^2 - \frac{x^3}{2!} + \frac{x^4}{3!} - \ldots$$

$$= 1 - 2x + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{5}{24}x^4 - \dots .$$

The rth term will be $(-1)^{r-1}x^{r-1}\left\{\frac{1}{(r-1)!} + \frac{1}{(r-2)!}\right\}$

$$= (-1)^{r-1}x^{r-1} \,.\, r/(r-1)!$$

The Hyperbolic Functions

Given the definitions

Hyperbolic Sine of $x \equiv \sinh x \equiv \frac{1}{2}(e^x - e^{-x})$,
Hyperbolic Cosine of $x \equiv \cosh x \equiv \frac{1}{2}(e^x + e^{-x})$,

(i) write out the series for $\sinh x$ and for $\cosh x$.
(ii) prove from the definitions that $\cosh^2 x - \sinh^2 x = 1$.

(i) Writing down $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots ,$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots ,$$

we see that $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2r+1}}{(2r+1)!} + \dots ,$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2r}}{(2r)!} + \dots .$$

(ii) $\cosh^2 x$ means the square of $\cosh x$, as $\cos^2 x$ is the square of $\cos x$.

By the definitions $\cosh^2 x = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$,

$$\sinh^2 x = \frac{1}{4}(e^{2x} - 2 + e^{-2x});$$

$$\therefore\ \cosh^2 x - \sinh^2 x = \frac{1}{4}(2 + 2) = 1.$$

Alternatively, $\cosh x + \sinh x = e^x$ and $\cosh x - \sinh x = e^{-x}$,

so $(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x \times e^{-x} = 1.$

Note that just as $(a \cos t, a \sin t)$ necessarily is a point on the circle $x^2 + y^2 = a^2$, so the point $(a \cosh t, a \sinh t)$ is necessarily a point on the (rectangular) hyperbola $x^2 - y^2 = a^2$. Hence the "hyperbolic" in the names.

There is one important difference, however; to every point on the circle $x^2 + y^2 = a^2$ there corresponds a value of t, but since $\cosh t$ is positive for all values of t only points having positive x can be determined by $(a \cosh t, a \sinh t)$.

It is usual to pronounce cosh in the natural way as spelt, and most people pronounce sinh as sinsh to match cosh.

Examples 94

Write down the first four terms and the $(n+1)$th term of the following series :

1. Exp $2x$. 2. Exp $(-3x)$. 3. Exp $\left(-\frac{p}{q}\right)$.

4. Exp $(1)+\exp(-1)$. 5. $(1+x)\exp x$. 6. $(1-2x)\exp(-x)$.

7. Calculate the value of e from the series correct to 4 places of decimals.

8. Calculate the value of $e^{\frac{1}{10}}$ from the series correct to 12 places of decimals.

9. Find the value of cosh 2 to 3 places of decimals (i) from the series (ii) as $\frac{1}{2}(e^2+e^{-2})$.

10. What is the nth term of the series $\exp(\log_e 2)$? What is the sum of this series?

11. Find the first 4 terms and the coefficient of x^r in the expansions of (i) $(1-x^2)e^{\frac{1}{2}x}$, (ii) $(e^x+1)^2$.

12. Find the first 4 terms and the coefficient of x^r in the expansions of (i) $(p+qx)e^{3x}$, (ii) $p\cosh x+qx\sinh x$.

13. In the expansion of e^4 find which term is the first to be less than unity, showing it to be 2048/2520.

14. Which term in the expansion of $e^{7\frac{1}{2}}$ is such that every subsequent term is less than one-third of the previous one?

15. Defining $\tanh x$ (pronounced tansh x) as $\dfrac{\sinh x}{\cosh x}$, to match

$$\tan x=\frac{\sin x}{\cos x},$$

prove by division, using the first 3 terms of each series, that

$$\tanh x = x-\tfrac{1}{3}x^3+\tfrac{2}{15}x^5-\dots .$$

16. Use the definitions to prove that

$$\cosh 2x=\cosh^2 x+\sinh^2 x,$$
$$\sinh 2x=2\cosh x\sinh x.$$

17. Write down the first six terms of $\exp x$ and of $\exp y$ and multiply them together.

Show that up to terms of the fifth degree in x and y together, the result is the first six terms of $\exp(x+y)$.

[If this process is carried on—and it can be proved that it is legitimate to do so—we shall be proving *from the series* that

$$\exp x\times\exp y=\exp(x+y)$$

without any appeal to the index laws for e^x.]

The Logarithmic Series

Just as we go from the series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \quad \dots\dots\dots\dots\text{(i)}$$

to the series

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots \quad \dots\dots\dots\dots\text{(ii)}$$

by differentiating the series term by term,
so we go back from series (ii) to series (i) by integrating term by term.

This is done by taking the definite integrals of the separate terms.

Now if $|t| < 1$, $1/(1+t) = 1 - t + t^2 - t^3 + \dots + (-1)^n t^n + \dots$.

Integrate each side of this from 0 to x when $|x| < 1$.

The L.H.S. gives $\log_e(1+x) - \log_e(1+0)$ which $= \log_e(1+x)$.

The R.H.S. gives the series

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots + (-1)^n x^{n+1}/(n+1) + \dots;$$

$$\therefore \log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots + (-1)^n \frac{x^{n+1}}{n+1} + \dots \text{ if } |x| < 1.$$

This series is *the logarithmic series.*

In this, $|x|$ must be less than 1, but x may be negative.

If we write $-x$ for x, the result is

$$\log_e(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^{n+1}}{n+1} + \dots\right) \text{ if } |x| < 1.$$

The negative result, when x is positive, agrees with the fact that the logarithm of a proper fraction is necessarily negative.

It is often convenient to omit the e in $\log_e x$ or $\log_e(1+x)$ when it is clear that logarithms to base e are being used and there is no danger of ambiguity.

Notice that there cannot be a series for $\log x$ in powers of x, for if we were to suppose

$$\log x = a_0 + a_1x + a_2x^2 + \dots,$$

putting $x = 0$ we should get $\quad -\infty = a,$

and so the finding of the coefficients breaks down at the outset.

Note carefully that the logarithmic series is like the binomial series in that it is only * convergent if $|x| < 1$. It is unlike the exponential series, which is convergent for all values of x.

* Except that it is convergent when $x = +1$ (not when $x = -1$).

Unless x is quite small the logarithmic series converges too slowly to be suitable for numerical calculation of logarithms.

But from it, another series more suitable for calculation is found thus:

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots$$

and

$$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\dots;$$

$$\text{Now } \log\frac{1+x}{1-x}\equiv\log(1+x)-\log(1-x),$$

$$\therefore\ \log\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right).$$

Example I. Put $x=\frac{1}{3}$ in the series for $\log\frac{1+x}{1-x}$ and calculate log 2 to 4 places. [Here log 2 means, of course, $\log_e 2$.]

If $x=\frac{1}{3}$, $\frac{1+x}{1-x}=2$ and $\log 2=2\left(\frac{1}{3}+\frac{1}{3^3\cdot 3}+\frac{1}{3^5\cdot 5}+\dots\right)$:

$\frac{1}{3}=\cdot 333{,}333,\dots$	1st term $=\cdot 333{,}333,\dots$
$\frac{1}{3^3}=\cdot 037{,}037,\dots$	2nd term $=\cdot 012{,}346,\dots$
$\frac{1}{3^5}=\cdot 004{,}115,\dots$	3rd term $=\cdot 000{,}823,\dots$
$\frac{1}{3^7}=\cdot 000{,}457,\dots$	4th term $=\cdot 000{,}065,\dots$
$\frac{1}{3^9}=\cdot 000{,}051,\dots$	5th term $=\cdot 000{,}006,\dots$
	Total $=\cdot 346{,}573$

$\therefore$ $\log 2=\cdot 6931$ to 4 places.

Note. Putting $x=1$ in the series for $\log(1+x)$ gives a convergent series equal to log 2, though neither the convergence nor the equality have been proved. However, the series converges too slowly to be of use for numerical calculation of log 2.

Example II. Put $\frac{1+x}{1-x}=\frac{n+1}{n}$ and deduce a series for $\log\frac{n+1}{n}$.

We get $n+nx=n+1-(n+1)x$ giving $x=1/(2n+1)$.

$$\therefore\ \log\frac{n+1}{n}=2\left\{\frac{1}{2n+1}+\frac{1}{3}\left(\frac{1}{2n+1}\right)^3+\frac{1}{5}\left(\frac{1}{2n+1}\right)^5+\dots\right\}.$$

Example III. Find a series in powers of x for $\log \dfrac{1+2x}{1-3x}$, stating the range for which the series is convergent.

We have $\quad \log(1+2x) = 2x - \dfrac{2^2x^2}{2} + \dfrac{2^3x^3}{3} - \dfrac{2^4x^4}{4} + \dots \quad |2x| < 1$

and $\quad \log(1-3x) = -3x - \dfrac{3^2x^2}{2} - \dfrac{3^3x^3}{3} - \dfrac{3^4x^4}{4} - \dots \quad |3x| < 1.$

To make both series convergent it is necessary that $|x| < \frac{1}{3}$. Then

$$\log\left(\frac{1+2x}{1-3x}\right) = \log(1+2x) - (1-3x)$$
$$= 5x + \tfrac{1}{2}(3^2 - 2^2)x^2 + \tfrac{1}{3}(3^3 + 2^3)x^3 + \tfrac{1}{4}(3^4 - 2^4)x^4 + \dots,$$

the nth term being $\dfrac{1}{n}\{3^n - (-1)^n 2^n\}x^n$.

Examples 95

1. Write down series for (i) $\log(1{\cdot}02)$, (ii) $\log({\cdot}99)$, (iii) $\log(1 + \frac{1}{2}x^2)$.

2. (i) In the series for $\log \dfrac{1+x}{1-x}$ put $x = \frac{1}{2}$ and calculate $\log 3$ from it to 4 places of decimals.

   (ii) In the series for $\log \dfrac{n+1}{n}$ put $n = 4$ and calculate $\log \frac{5}{4}$ from it to 5 places of decimals.

   Using the result of worked Example I, find $\log_e 5$ to 4 places.

3. Obtain $\log_e 10$ from $\log_e 2$ and $\log_e 5$ (Use result of 2 (ii)).

   How can this be used to find the common logarithms of 2, 3, 5, etc.? Use it to find $\log_{10} 2$ to 3 places.

4. What are the values of x so that $\dfrac{1+x}{1-x}$ may be (i) 5, (ii) 7, (iii) 11?

   (The series obtained for log 5, etc., converge too slowly to be useful for calculation.)

   Give series in ascending powers of x for the following Nos. 5 to 9, and give in each case the general term as well as the first 3 terms.

5. $\log(1+2x)$. 6. $\log(1+x)^2$. 7. $\log(1+x^2)$.

8. $\log\{(1+x)(1-2x)\}$. 9. $\log\{(1+x)(1+3x)\}$.

10. Expand $\log(2+x)$ as $\log 2 + \log\left(1 + \dfrac{x}{2}\right)$ and hence calculate $\log 2{\cdot}05$ given $\log 2 = {\cdot}6931$.

11. Prove that:

    (i) $\log x = 2\left\{\dfrac{x-1}{x+1} + \dfrac{1}{3}\cdot\dfrac{(x-1)^3}{(x+1)^3} + \dfrac{1}{5}\dfrac{(x-1)^5}{(x+1)^5} + \dots\right\}$;

    (ii) $\log(1+x) = \log x + 2\left\{\dfrac{1}{2x+1} + \dfrac{1}{3}\cdot\dfrac{1}{(2x+1)^3} + \dfrac{1}{5}\cdot\dfrac{1}{(2x+1)^5} + \dots\right\}$;

(iii) $\log(x+1)=2\log x-\log(x-1)$

$$-2\left\{\frac{1}{2x^2-1}+\frac{1}{3}\frac{1}{(2x^2-1)^3}+\frac{1}{5}\frac{1}{(2x^2-1)^5}+\dots\right\}.$$

12. If $|x|>1$ prove that $\log(x+1)-\log x=\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{x^2}+\frac{1}{3}\cdot\frac{1}{x^3}+\dots.$

13. Putting $x=0\cdot1$ in the series for $\log(1+x)$ find $\log 11/10$ to 5 places.

Find this also from the series for $\log\frac{n+1}{n}$ (Example II).

14. If $P=-\log(1+\frac{1}{10})$, $Q=-\log(1-\frac{1}{25})$, $R=\log(1+\frac{1}{80})$, prove that:

$$\begin{aligned}\log 2&=\ \ 7P-2Q+3R,\\ \log 3&=11P-3Q+5R,\\ \log 5&=16P-4Q+7R.\end{aligned}$$

Calculate the logarithms to base e of 2, 3, 5 and 10 from these results, to 5 figures.

Deduce the logarithms to base 10 of 2, 3 and 5.

15. If $a+b+c=0$, $ab+bc+ca=q$, $abc=r$, show that

$$\log(1-ax)+\log(1-bx)+\log(1-cx)\equiv\log(1+qx^2-rx^3).$$

Using the logarithmic series, and by equating the coefficients, show that:

(i) $a^2+b^2+c^2=-2q$;
(ii) $a^4+b^4+c^4=2q^2$;
(iii) $a^6+b^6+c^6=-2q^3+3r^2$;
(iv) $6\Sigma a^5=5\,\Sigma a^3\,.\,\Sigma a^2$.

The Trigonometric Series

That there are expansions in powers of x (the radian measure) for $\sin x$ and $\cos x$ is suggested by some approximate values when x is small, which may have been learnt earlier.

Since $\underset{x\to 0}{\mathrm{Lt}}\frac{\sin x}{x}=1$, it follows that if x is small $\sin x\simeq x$.

Hence $\cos x\equiv 1-2\sin^2\frac{x}{2}\simeq 1-2\frac{x^2}{4}$, i.e. $\cos x\simeq 1-\frac{x^2}{2}$.

From these results further approximations can be obtained by integration; thus $\sin x=\int\cos x\,dx\simeq x-\frac{x^3}{2\cdot 3}+k$ where $x=0$ gives $k=0$ and $\cos x=-\int\sin x\,dx\simeq-\frac{x^2}{2}+\frac{x^4}{2\cdot3\cdot4}+k$ where $x=0$ gives $k=1$.

But these results are only useful when x is quite small. Also the error in them is uncertain.

So it is better to use inequalities * and the fact that the integral of a function >0 between the limits is >0, as follows.

Taking x and t positive and using $\cos t \leqslant 1$ when $t \geqslant 0$,

$$\int_0^x (1-\cos t)\,dt > 0; \quad \therefore\ x - \sin x > 0 \text{ or } \sin x < x.$$

$$\int_0^x (t-\sin t)\,dt > 0; \quad \therefore\ \frac{x^2}{2} + \cos x - 1 > 0, \quad \cos x > 1 - \frac{x^2}{2}.$$

$$\int_0^x \left(\cos t - 1 + \frac{t^2}{2}\right) dt > 0; \quad \therefore\ \sin x - x + \frac{x^3}{3!} > 0, \quad \sin x > x - \frac{x^3}{3!}.$$

$$\int_0^x \left(\sin t - t + \frac{t^3}{3}\right) dt > 0; \quad \therefore\ 1 - \cos x - \frac{x^2}{2} + \frac{x^4}{4!} > 0,$$

$$\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

and so on as far as we please, the results being

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots.$$

These results can also be obtained from the differential equations satisfied by $\sin x$ and $\cos x$, but not so easily as for the binomial and exponential series.

Since $\quad \dfrac{d}{dx}\sin x = \cos x \quad$ and $\quad \dfrac{d}{dx}\cos x = -\sin x,$

both $\sin x$ and $\cos x$ satisfy the equation

$$\frac{d^2y}{dx^2} = -y,$$

and hence this equation is satisfied by $A \sin x + B \cos x$ where A and B are any constants.

To ensure that the series for $\sin x$ and that for $\cos x$ are obtained separately we must make use of the facts that

$$\sin x = -\sin(-x) \text{ but } \cos x = +\cos(-x).$$

From these equations it follows that

the series for $\sin x$ can contain only odd powers of x

and the series for $\cos x$ can contain only even powers of x.

* As in the *Report on the Teaching of Calculus in Schools.*

If both even and odd powers occur the change from x to $-x$ will change the numerical value.

If now we assume

$$y=\sin x=a_1x+a_3x^3+a_5x^5+a_7x^7+\ldots \quad \text{(i)}$$

we get
$$\frac{dy}{dx}=\cos x=a_1+3a_3x^2+5a_5x^4+7a_7x^6+\ldots, \quad \text{(ii)}$$

$$\frac{d^2y}{dx^2}=-\sin x=6a_3x+20a_5x^3+42a_7x^5+\ldots. \quad \text{(iii)}$$

Comparing (i) and (iii), using $\dfrac{d^2y}{dx^2}=-y$, we get

$$-6a_3=a_1 \quad -20a_5=a_3 \quad -42a_7=a_5, \text{ etc.}$$

Giving
$$\sin x=a_1\left(x-\frac{x^3}{6}+\frac{x^5}{6\,.\,20}-\frac{x^7}{6\,.\,20\,.\,42}+\ldots\right)$$
$$=a_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right).$$

Since $\sin x \simeq x$ for small x and letting $x\to 0$, we see that $a_1=1$ and get the required result.

In the course of this process we have also found the series for $\cos x$, series (ii),

$$\cos x=a_1\left(1-\frac{3x^2}{3!}+\frac{5x^4}{5!}-\ldots\right);$$
$$\therefore \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots.$$

Convergence of these series

It will be seen that these series are like the exponential series in having x^n divided by $n!$

They are in fact obtained by taking either the odd or even powers of x in the exponential series and then changing every second sign to minus.*

It follows that, *like the exponential series, they are convergent for all values of x.*

Periodicity

The series discussed previously have, for positive values of x, increased as x increases, or decreased as x increases, as in the cases of $\log(1+x)$ and $\exp(-x)$. This seems the natural state of affairs, but it does not occur in the case of the series for $\sin x$ and $\cos x$.

* The explanation of this remarkable fact is given on p. 277.

Like the function represented, the series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ is zero when $x=0$, increases to 1 when $x=\frac{\pi}{2}$ and decreases to 0 again when $x=\pi$, after which it is negative till $x=2\pi$; also the value is the same for $x=a$ and $x=2\pi+a$.

Again the series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ diminishes from 1 to 0 as x increases from 0 to $\frac{\pi}{2}$, then becomes negative and is such that its values for $x=a$ and $x=2\pi-a$ are the same.

These results have been proved when we have proved that the series represent the functions $\sin x$ and $\cos x$.

Approximate numerical verification is possible, but takes a good deal of work. (See Example IV below.)

Example I. Obtain series in powers of x for $\sin 2x$ and $\sin^2 x$.

(i) Replacing x by $2x$ in the series for $\sin x$, we have

$$\sin 2x = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \ldots;$$

(ii) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$= \tfrac{1}{2}\left\{1 - \left(1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \ldots\right)\right\}$$

$$= \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \ldots.$$

Example II. Calculate the sine of one-fifth of a radian from the series:

$$\sin\frac{1}{5} = \frac{1}{5} - \frac{\frac{1}{5^3}}{3!} + \frac{\frac{1}{5^5}}{5!} - \ldots.$$

The calculation is as follows:

1st term		=	·200,000,000
Divide by	5		
Divide by	5		[·040,000,000]
Divide	6		[·008,000,000]
2nd term		=	·001,333,333
Divide by	5		
Divide by	5		[·000,266,666]
Divide by	20		[·000,053,333]

3rd term = ·000,002,666
Divide by 5
Divide by 5 ·000,000,533
Divide by 6 ·000,000,106
Divide by 7 ·000,000,017
4th term = ·000,000,002

Adding 1st and 3rd terms and subtracting 2nd and 4th terms:

$$\sin \tfrac{1}{5} = \cdot 200{,}002{,}666$$
$$- \cdot 001{,}333{,}335$$
$$= \cdot 198{,}669{,}331.$$

Example III. Using terms of the series for $\sin x$ and for $\cos x$ as far as x^8, work out $\sin^2 x + \cos^2 x$ as far as the term in x^8.

$$\sin^2 x = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots\right)^2$$
$$= x^2 - \frac{x^4}{3} + x^6\left(\frac{1}{36} + \frac{1}{60}\right) - x^8\left(\frac{1}{2520} + \frac{1}{360}\right) + \dots$$
$$= x^2 - \frac{x^4}{3} + x^6\frac{16}{360} - x^8\frac{8}{2520}$$
$$\cos^2 x = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \dots\right)^2$$
$$= 1 - x^2 + x^4\left(\frac{1}{4} + \frac{1}{12}\right) - x^6\left(\frac{1}{360} + \frac{1}{24}\right) + x^8\left(\frac{1}{576} + \frac{1}{720} + \frac{1}{20160}\right)$$
$$= 1 - x^2 + x^4 \,.\, \frac{1}{3} - x^6\frac{16}{360} + x^8\frac{64}{20160} + \dots,$$

but $\sin^2 x \quad = \quad x^2 - x^4 \,.\, \frac{1}{3} + x^6\frac{16}{360} - x^8\frac{8}{2520} + \dots;$

$$\therefore \sin^2 x + \cos^2 x = 1 + 0x^2 + 0x^4 + 0x^6 + 0x^8 + \dots.$$

[By working out the general terms, it can be proved that

$$\sin^2 x + \cos^2 x = 1$$

from the series.]

Example IV. Using 4 terms of the cosine series and taking $\pi^2 \simeq 10$ obtain a rough approximation to $\cos\frac{\pi}{2}$.

The terms are $\quad 1 - \dfrac{\pi^2}{4 \,.\, 2!} + \dfrac{\pi^4}{16 \,.\, 4!} - \dfrac{\pi^6}{64 \,.\, 6!}$

$$\simeq \pi^6\left\{\frac{1}{1000} - \frac{1}{800} + \frac{1}{3840} - \frac{1}{46080}\right\}$$
$$\simeq 1000\{\cdot 001 - \cdot 00125 + \cdot 00026 - \cdot 00002\}$$
$$= 1000\{\cdot 00126 - \cdot 00127\}$$
$$= -1000 \times \cdot 00001 = -\cdot 01, \text{ and the fifth term is positive.}$$

[A more accurate value for π^2 and a few more terms would bring the result nearer to zero.]

Gregory's Series for $\tan^{-1} x$

Just as the logarithmic series for $\log(1+x)$ was obtained by integrating both sides of the identity

$$\frac{1}{1+t}=1-t+t^2-t^3+\dots \qquad \text{if } |t|<1,$$

so another important series is obtained by integrating both sides of the identity

$$\frac{1}{1+t^2}=1-t^2+t^4-t^6+\dots \qquad \text{if } |t|<1.$$

If $y=\tan^{-1} t$, it can be shown that $\dfrac{dy}{dt}=\dfrac{1}{1+t^2}$, so that

$$\int_0^x \frac{dt}{1+t^2}=\tan^{-1} x-\tan^{-1} 0.$$

Hence integrating the second identity above from 0 to x we get if $|x|<1$,

$$\tan^{-1} x-\tan^{-1} 0=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots.$$

Now if $\theta=\tan^{-1} x$, $|\tan\theta|<1$ if θ lies between $-\dfrac{\pi}{4}$ and $+\dfrac{\pi}{4}$ or between $\dfrac{3\pi}{4}$ and $\dfrac{5\pi}{4}$ or between $\dfrac{7\pi}{4}$ and $\dfrac{9\pi}{4}$ or, etc.

In the range $-\dfrac{\pi}{4}$ to $+\dfrac{\pi}{4}$ the angle $\tan^{-1} 0$ is itself 0, so in this range we get

$$\tan^{-1} x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$$

or

$$\theta=\tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\frac{\tan^7\theta}{7}+\dots. \qquad \text{(I)}$$

Either of these statements is what is called Gregory's Series.

Note that in the range $\dfrac{3\pi}{4}$ to $\dfrac{5\pi}{4}$ the angle $\tan^{-1} 0$ is π, and the left-hand side of (I) would for this range be $\tan^{-1} x-\pi$.

Calculation of π

Using the formula for $\tan(A+B)$, we see that

$$\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\tan^{-1} 1=\frac{\pi}{4}.$$

Hence the value of $\frac{\pi}{4}$ is found as the sum of the two series

$$\tfrac{1}{2} - \tfrac{1}{3}(\tfrac{1}{2})^3 + \tfrac{1}{5}(\tfrac{1}{2})^5 - \tfrac{1}{7}(\tfrac{1}{2})^7 + \ldots$$

and

$$\tfrac{1}{3} - \tfrac{1}{3}(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 - \tfrac{1}{7}(\tfrac{1}{3})^7 + \ldots$$

from which the value of π can readily be found to a few decimal places.

Note: The Logarithmic Series and Gregory's Series are each convergent provided $|x| < 1$, and in that range give the values of $\log(1+x)$ and $\tan^{-1} x$ respectively.

It is shown in Chapter XII that the first of these series is convergent if $x = +1$, but divergent if $x = -1$; the second behaves similarly.

It can be proved, but the proof is outside the work covered in this book, that when $x = +1$ the series still give the values of the functions, which are given by the series when $|x| < 1$, so that

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots$$

and

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \ldots .$$

Both these series, however, converge so slowly as to be useless for purposes of calculation, and other series for calculating log 2 and $\frac{\pi}{4}$ have been given already.

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1. Use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to obtain a series for $\cos^2 x$, stating the first four terms.
2. Calculate from the series:
   (i) Sine of one-tenth of a radian to 8 places.
   (ii) Cosine of one radian to 4 places.
3. Using four terms of each series find a series for $2 \sin x \cos x$ as far as x^7, comparing the result with the series for $\sin 2x$.
4. As in worked Example IV find a rough approximation (from the series) for $\frac{\sin \pi}{\pi}$.
5. If x is small, find the first three terms of a series for $\tan x$ in ascending powers of x by dividing $\sin x$ by $\cos x$, taking each series to 4 terms.
6. Show that the first two terms in the series for $\sin\left(\frac{\pi}{30}\right)$ give its value

correct to 4 decimal places by considering the value of the third term.

Find the value of sin 6° given by these first two terms and compare with the value given in the tables.

7. Use the series given on p. 252 to calculate the value of π to four decimal places.

Find also the values of $\tan^{-1} \frac{1}{2}$ and $\tan^{-1} \frac{1}{3}$ separately and compare with the radian measure of these angles as found in the tables.

8. Find Gregory's Series by assuming

$$y = \tan^{-1} x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

and substituting in the differential equation

$$(1 + x^2) \frac{dy}{dx} = 1.$$

9. (i) Show that $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$.

(ii) Use the series for these angles to calculate the value of π.

10. Obtain a power series for $\sin^{-1} x$ from the differential equation $\frac{d}{dx}(\sin^{-1} x) = (1 - x^2)^{-\frac{1}{2}}$.

11. Use the series for e^{-t} and $\sin t$ to find the first four terms in the power series for $e^{-t} \sin t$.

12. Prove Machin's formula $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \frac{\pi}{4}$, and write down an expression (in series) equal to π.

13. If $\tan \theta$ is so small that $\tan^7 \theta$ may be neglected, use the equation

$$\theta = \tan \theta - \frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5}$$

to find, in succession, the approximations

$$\tan \theta = \theta; \quad \tan \theta \simeq \theta + \frac{\theta^3}{3},$$

and find the next approximation.

14. By multiplying together the series for $\sin x$, $\cosh x$, etc., show that $\sin x \cosh x - \cos x \sinh x = \frac{2}{3} x^3 +$ terms of higher power than x^6.

Maclaurin's Expansion

If for any function $f(x)$ of x there is an equivalent convergent power series, a general form for each coefficient in that series can be found by repeated differentiation.

Thus if $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\dots+a_nx^n+\dots,$

then

$$f'(x)=a_1+2a_2x+3a_3x^2+\dots+n\,.\,a_nx^{n-1}+\dots,$$

$$f''(x)=2a_2+3\,.\,2a_3x+4\,.\,3a_4x^2+\dots+n(n-1)a_nx^{n-2}+\dots,$$

$$f'''(x)=3\,.\,2a_3+4\,.\,3\,.\,2a_4x+\dots+n(n-1)(n-2)a_nx^{n-3}+\dots,$$

and so on.

In these equations put $x=0$, remembering that, for example, $f''(0)$ means the value of $f''(x)$ if x is put equal to 0 *after* the differentiation has been performed; we get

$f(0)=a_0$	so that	$a_0=f(0)$,
$f'(0)=a_1$		$a_1=f'(0)$,
$f''(0)=2a_2$		$a_2=\frac{1}{2!}f''(0)$,
$f'''(0)=3\,.\,2a_3$		$a_3=\frac{1}{3!}f'''(0)$,
and so on.		and so on.

The general result for a_n being $a_n=\frac{1}{n!}f^n(0)$.

In all this it has been assumed that for any values of x concerned the values of $f(x)$, $f'(x)$, $f''(x)$, etc., are finite.

$\therefore$ we have, in general,

$$f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\frac{1}{3!}f'''(0)x^3+\dots+\frac{1}{n!}f^n(0)x^n+\dots\quad\text{(I)}$$

This series is known as *Maclaurin's Expansion.*

All the power series discussed in this chapter can be found by using this general result. It has already been used for two of them.

Another result, known as Taylor's Expansion, of which the above is a particular case, can be proved in the same way, *if the initial assumption is made that the series exists.* This is

$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{2!}f''(a)+\dots+\frac{x^n}{n!}f^n(a)+\dots\quad\text{(II)}$$

Example I. Use Maclaurin's Expansion to obtain the series for $\log(1+x)$.

If $f(x)=\log(1+x)$, $\quad f(0)=\log 1=0$,

$f'(x)=\frac{1}{1+x}$, $\quad f'(0)=1$,

$$f''(x) = -\frac{1}{(1+x)^2}, \qquad f''(0) = -1,$$

$$f'''(x) = +\frac{2}{(1+x)^3}, \qquad f'''(0) = 2,$$

$$f^{\mathrm{IV}}(x) = -\frac{3 \cdot 2}{(1+x)^4}, \qquad f^{\mathrm{IV}}(0) = -3!,$$

etc.

$\therefore$ the series is

$$\log(1+x) = 0 + x - \frac{1}{2}x^2 + \frac{2}{3!}x^3 - \frac{3!}{4!}x^4 + \dots,$$

which is the usual series.

Examples 97

1. Obtain by Maclaurin's Expansion the series for

   (i) e^{2x}; (ii) $\sin x$; (iii) $\cos 2x$; (iv) $\tan^{-1} x$.

2. Assume that $f(a+x) \equiv a_0 + a_1 + a_2x^2 + \dots$, and prove that (II), i.e. Taylor's Expansion, gives the value of the coefficients.
3. Use Taylor's Theorem to obtain the series for $(1+x)^m$ and for $(1-x)^{-m}$.
4. If $f(x) = a^x$ find $f'(x)$, $f''(x)$, $f'''(x)$, etc., and hence the series for $f(x)$.
5. Find the first four terms in the expansion of $e^{-x} \sin x$ using Maclaurin's Expansion. (Compare No. 11 of Examples 96.)
6. Show that $e^x(\cos x + \sin x) = (1+x)^2$ if terms higher than the third degree are neglected.
7. Establish the following:

$$\text{(i)}\ \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots;$$

$$\text{(ii)}\ \frac{x}{\tan x} = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \dots;$$

$$\text{(iii)}\ \log(1+\sin x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \dots.$$

8. Show that an attempt to obtain an equivalent convergent series by Maclaurin's Expansion fails in the case of (i) $\sqrt{x}$, (ii) $\log x$.

Note. In order to prove the Expansions I and II without the initial assumption *that the series exists*, it is necessary to find a formula for the remainder after n terms on the right so that it may be equal to the function on the left. For this the student is referred to books on the Calculus.

For further examples of the series and functions discussed in this chapter, see the set of examples placed after Chapter XII.

Test Papers B

B.I

1. Define a logarithm, and from your definition prove that

$$\log_x \frac{a^n}{b} = n \log_x a - \log_x b.$$

Use logarithms to evaluate $\dfrac{10^{\frac{2}{3}}(3\,.\,416^2 - 2)}{\sqrt[3]{\cdot 0217}}$. (L.)

2. A man contributes £75 per annum to a pension fund in which money accumulates at 3% per annum compound interest. What is the amount to his credit in the fund at the end of 35 years, the payments being made at the beginning of each year?

3. Write down the expansion of $(1 + x)^7$ by the Binomial Theorem, giving each coefficient in its numerical form.

Prove that
$$\frac{(1 + \cdot 002)^7 (1 - \cdot 002)^8}{(1 \cdot 0003)^{11}}$$
is approximately $\cdot 9947$.

To how many significant figures is this result correct? (L.)

4. Sum the geometric progression $72 + 48 + 32 + \ldots$ to infinity and express the recurring decimal $\cdot 6\dot{7}\dot{2}$ as a vulgar fraction by using the rule for summing a geometric progression. (L.)

5. If in the quadratic equation $ax^2 + bx + c = 0$, the coefficient b is much larger numerically than either a or c, show that one root is large and that
$$-\frac{c}{b} - \frac{ac^2}{b^3}$$
is an approximation to the other.

6. By drawing the curves $y = \dfrac{1}{x}$ and $4y = (x - 2)^2$, show that the equation $x(x - 2)^2 = 4$ has one real root just greater than 3 and find whether the root is greater or less than $3 \cdot 2$. (L.)

B.II

1. Solve the equations
$$xy = -\tfrac{8}{9}, \quad x^2 + xy + y^2 = \tfrac{4}{3}. \qquad \text{(B.)}$$

2. (i) Write down the ratio of the coefficient of x^r to the coefficient of x^{r-1} in the expansion of $(1 + x)^n$, and show that this ratio is $\frac{1}{2}$ if $2n = 3r - 2$.

(ii) If the coefficients of three consecutive terms of $(1 + x)^n$ are in the ratio 6 : 3 : 1, find n. (B.)

3. If a, b, c are the first three terms of an arithmetical progression and also the first, third and fourth terms of a geometrical progression, prove that

(i) $a(a^2+4b^2)=b(b^2+4a^2)$;

(ii) $b=\frac{1}{2}a(3\pm\sqrt{5})$. (B.)

4. (i) If

$$\frac{x^2+2x+5}{(x-1)^2(x-3)}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x-3}$$

for all values of x, find the constants A, B, C.

(ii) Find the factors of

$$x^3-5x^2+7x-3. \qquad \text{(B.)}$$

5. If the digits 1, 2, 3, 4, 5, 6, 7, 8 are written down at random in a row, find the chance that no two odd integers are next to one another.

6. Define e as a series and show that it is an incommensurable number.

Prove that

$$e+3=1+\frac{5}{2!}+\frac{9}{3!}+\ldots+\frac{4n-3}{n!}+\ldots. \qquad \text{(B.)}$$

B.III

1. (i) Solve the equations

$$x^2+3xy+2y^2-x-8=0,$$
$$2x+3y=6.$$

(ii) Find the sum of all even integers from 2 to 100 inclusive, excluding those which are multiples of 3. (L.)

2. If s is the sum to n terms of an A.P. of first term a and common difference d, show that

$$n^2d-n(d-2a)-2s=0.$$

Hence, if a and d are integers, prove that $(d-2a)^2+8sd$ is a perfect square.

If $s=247$, $a=1$ and $d=3$, verify that this is so and find n. (L.)

3. If the letters stand for positive numbers and if $a+b=c$ prove that ab is not greater than $\frac{c^2}{4}$.

If $a+b=c$ prove that $(2a-b)(a-2b)=2c^2-9ab$ and deduce that if ab is not greater than $\frac{2c^2}{9}$, then one of the numbers a, b must be at least double the other. (L.)

4. Sum the series $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\ldots$ to n terms.

What is the sum to infinity?

Show that the sum to 1000 terms differs from the sum to infinity by less than $\frac{1}{1000}$.

5. How many terms are there in the expansion of $(a+b+c)^{10}$?

Prove that the number of terms that contain a is 5 times the number of terms that do not. (L.)

6. Determine b and c so that the coefficients of x^3 and x^4 in the expansion of $(1+bx+cx^2)\log_e(1+x)$ may vanish.

Prove that with these values of b and c the error in taking

$$\frac{x+\frac{1}{2}x^2}{1+bx+cx^2}$$

for $\log_e(1+x)$ is $\frac{x^5}{180}$, neglecting powers of x higher than the fifth. (L.)

B.IV

1. If $f(x)\equiv ax^2+2bx+c$, where $a\neq 0$, prove that

$$f(x)\,.\,f(y)\equiv(ac-b^2)(x-y)^2$$

if, and only if, x and y are connected by the equation

$$axy+b(x+y)+c=0.$$

Examine the special case $ac=b^2$. Is the statement also true if $a=0$?

2. Show graphically that the equation $e^x=\frac{x+1}{x}$ has two real roots and find the positive root correct to 3 decimal places. (B.)

3. By considering the number of ways of choosing r letters from a_1, a_2, b_1, b_2, b_3 ... b_n prove from first principles that

$${}^{n+2}C_r={}^nC_r+2\,.\,{}^nC_{r-1}+{}^nC_{r-2}$$

without *assuming* any formula for ${}^nC_r$. (L.)

4. Express $\frac{3x^4-x^3+x^2-x+2}{(x-1)(x^2-1)(x^2+1)}$ as a sum of partial fractions in their simplest form.

Show that, if $-1<x<+1$, the function can be expanded in a series of ascending powers of x and that the coefficients of x^{2n} and x^{2n+1} are respectively

$$(2n+1)+(-1)^n,\quad 2n+(-1)^n.$$

Find the sum of the coefficients of the odd powers of x up to x^{4n-1}. (B.)

5. Find the sum of the series

$$1^2 - 2^2 + 3^2 - 4^2 + \dots - (2n)^2.$$

Find also the ratio of the sum of the odd terms in this series to the sum of the even terms with their signs changed.

6. (i) Assuming the logarithmic series, prove that

$$\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots \text{ ad. inf.} \quad \text{when } |x| < 1.$$

(ii) If $$S_1 = \frac{1}{7} + \frac{1}{3}\cdot\frac{1}{7^3} + \frac{1}{5}\cdot\frac{1}{7^5} + \dots \text{ ad. inf.}$$

and $$S_2 = \frac{1}{17} + \frac{1}{3}\cdot\frac{1}{17^3} + \frac{1}{5}\cdot\frac{1}{17^5} + \dots \text{ ad. inf.},$$

show that $\log_e 2 = 4S_1 + 2S_2$, and find $\log_e 3$ in terms of S_1 and S_2. (B.)

B.V

1. On two successive days pounds sterling were quoted in New York at \$$a$ to the pound and \$$b$ to the pound.

A man bought £x each day, paying in dollars; another man applied \$$y$ each day to the purchase of pounds. What average price in dollars to the pound was paid by each man? What are these means of a and b called? Which is the larger? (L.)

2. Factorise

(i) $x^6 - y^6$;

(ii) $2x^2 - xy - y^2 + 3x + 3y - 2$. (B.)

3. Explain the meaning of the statement $x = \log_{10} y$, using indices.

Show how logarithms may be used to solve for x an equation of the type $a^x = b^{x+c}$ and illustrate by solving $(5{\cdot}4)^x = (4{\cdot}5)^{x+1}$.

What is the sum to 20 turns of the series

$$\log\tfrac{2}{1} + \log\tfrac{3}{2} + \log\tfrac{4}{3} + \log\tfrac{5}{4} + \dots,$$

the logarithms being to base 10? (L.)

4. Solve the equations

(i) $\dfrac{3}{2(x+1)} + 2 + \dfrac{4x+3}{6} = \dfrac{x}{12}$;

(ii) $3x^2 - xy + 5x = 9,\ 3x + y = 8$. (B.)

5. Given that $(1+x)^{2n} = c_0 + c_1x + c_2x^2 + \dots + c_{2n}x^{2n}$, prove that

(i) $c_0 + c_1 + c_2 + \dots + c_{2n} = 2^{2n}$,

(ii) $c_0 + c_2 + c_4 + \dots + c_{2n} = 2^{2n-1}$.

If the coefficients of the third and fourth terms in the above expansion are in the ratio 3 : 8, find the value of n. (B.)

6. If x is small and if

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4},$$

show that we obtain as successive approximations for x in powers of y

$$x = y, \quad x = y + \frac{y^2}{2}, \quad x = y + \frac{y^2}{2} + \frac{y^3}{6},$$

and obtain the next approximation. (L.)

B.VI

1. At the end of any year a car is estimated to be worth $(1 - 1/q)$ times its value at the beginning of that year. Find the estimated value of a car at the end of n years if it cost £P originally.

   If the original value was £1,250 and the value after four years is £512, show that after a further year has elapsed its estimated value will be reduced to £409 12s. (L.)

2. Find the nth term u_n and the sum S_n of the first n terms of an arithmetical progression whose first term is a and common difference d.

   Show that the sum

$$u_n + u_{2n} + u_{3n} + \dots \text{ to } n \text{ terms}$$

   is equal to $na + \frac{1}{2}n(n-1)(n+2)d$. (L.)

3. (i) If $\dfrac{3(x-3)}{(x-1)(x+2)}$ is expressed in the form

$$\frac{A}{x-1} + \frac{B}{x+2},$$

   find A and B.

   (ii) A curve whose equation is $y = x^n + b$ passes through the points $(1, 7)$, $(3, 33)$; find n and b. (B.)

4. Obtain the coefficients in the expansion of $(a+b)^6$. If $a = 10b$, show that the third term in the expansion is $7\frac{1}{2}$ times the fourth term and find the ratio of the fourth term to the fifth. (B.)

5. Draw the graph of $y = \log_2 x$ for values of x between 0 and 8 taking 1 inch as the unit for both x and y. Also with the same axes and unit of length the graph of $y = x^2 - 3x$.

   Hence find approximate solutions of the equation

$$2^{x^2-3x} = x. \quad \text{(L.)}$$

6. Write down the expansion of $\log_e(1-x)$ in ascending powers of x, stating the range of values of x for which the expansion is valid.

If x is any positive number and $u=\dfrac{x}{1+x}$, show that

$$\log_e(1+x)=u+\frac{u^2}{2}+\frac{u^3}{3}+\dots,$$

and hence that

$$\log_e(1+x)>\frac{x}{1+x}.$$

(O. & C.)

B.VII

1. The two expressions $\frac{1}{6}\pi h\{a^2+b^2+(a+b^2)\}$ and $\frac{1}{3}\pi\{b^2(h+x)-a^2x\}$ are found by different methods for the volume of a certain solid. Prove that they are equivalent provided that

$$\frac{b}{h+x}=\frac{a}{x}. \qquad \text{(L.)}$$

2. Find a formula for the sum of n terms of an arithmetic progression whose first term is a and common difference d.

If the sum of the first and second terms of an arithmetic progression is x, and if the sum of the nth and $(n-1)$th terms is y, prove that the sum of the first n terms is $\dfrac{n(x+y)}{4}$ and that the common difference is $\dfrac{y-x}{2n-4}$. (L.)

3. If $x+y+z=0$ prove that $x^3+y^3+z^3=3xyz$. Hence show that, if

$$(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3=0$$

the three numbers a, b, c can be arranged in an arithmetical progression. (L.)

4. State and prove the Binomial Theorem for the expansion of $(1+x)^n$, where n is a positive integer.

If u_r denote the rth term of this expansion show that

$$\frac{u_r}{u_{r+1}}=\frac{r}{(n-r+1)x}.$$

Deduce that there is no such value of n for which three consecutive terms of this expansion form a geometric progression. (L.)

5. (i) If logarithms are to base 10 and $z=\log 2$ express in terms of z the logarithms of $\frac{125}{128}$ and of $\frac{625}{512}$ and deduce that $\frac{3}{10}<z<\frac{4}{13}$.

(ii) Prove that $a^{\log b}=b^{\log a}$ whatever the base of logarithms.

(L.)

6. Prove that the equation whose roots are the four numbers obtained by adding a root of $x^2 - 2bx + c = 0$ to a root of $x^2 - 2b'x + c' = 0$ is

$$\{x^2 - 2(b + b')x + c + c' + 2bb'\}^2 - 4(b^2 - c)(b'^2 - c') = 0.$$

(L.)

B.VIII

1. The symbol $[x]$ denotes the greatest integer not exceeding x. Write down the values of

$$[2\tfrac{1}{2}], \quad [2], \quad [-2\tfrac{1}{2}], \quad [-2].$$

Indicate the shapes of the graphs of

$$\text{(i) } y = [x], \quad \text{(ii) } y = x - [x], \quad \text{(iii) } y = [2^x].$$

2. (*a*) Solve the simultaneous equations

$$x + y = a,$$
$$x^2 + y^2 = 2a^2.$$

(*b*) Given that no two of a, b, c are equal, solve the simultaneous equations

$$a(b - y) + b(a - x) = (a - x)(b - y),$$
$$a^2(b - y) + b^2(a - x) = c(a - x)(b - y).$$

3. The common ratio of successive terms of a geometric series is $\dfrac{6r}{r^2 - r + 6}$. Find the range of values of r for which the series may be summed to infinity.

4. If x, y, z, w are consecutive terms of an arithmetical progression, prove that constants b, c, d can be found such that

$$x^2 + by^2 + cz^2 + dw^2 = 0$$

and find their values.

5. Use the binomial expansion to calculate the value of $\left(1 + \dfrac{1}{1000}\right)^{200}$ correct to four decimal places.

6. Find the sums of the series

(i) $1^2 + 2^2 + 3^2 + \ldots + n^2$.

(ii) $n \cdot 1^2 + (n - 1) \cdot 2^2 + (n - 2) \cdot 3^2 + \ldots + 1 \cdot n^2$.

Find also the sum of the infinite series

$$1 + x + 2x^2 + \ldots + \frac{n^2 x^n}{n!} + \ldots$$

CHAPTER XI

COMPLEX NUMBERS

CONSIDER the three quadratic equations :

$$x^2-6x+5=0, \quad x^2-6x+9=0, \quad x^2-6x+13=0.$$

The first can be written $(x-3)^2=4$ and gives the two solutions or roots $x-3=\pm 2$ or $x=3\pm 2$, that is $x=5$ or 1.

The second can be written $(x-3)^2=0$, so there is only one root $x=3$; but to make this equation match the first one, and as there are two equal factors, it is customary to say that there are *two equal roots.*

The third can be written $(x-3)^2=-4$, so there are no roots, for there is no number whose square is -4.

Now mathematicians dislike saying "a quadratic equation sometimes has two roots and sometimes no roots", so a symbol is introduced, i, to stand for $\sqrt{-1}$ such that $i^2=-1$, but otherwise i is to obey the ordinary laws of algebra.

It follows that $(2i)^2=-4$, and also $(-2i)^2=-4$.

So the equation $(x-3)^2=-4$ gives $x-3=2i$ or $-2i$, and hence $x=3+2i$ or $x=3-2i$; two roots as in the other cases.

There is, however, this obvious difference :

If $x=3+2$ then $x=5$ and *one* number 5 gives the root; but if $x=3+2i$ the *two* numbers 3 and 2 must be kept separate.

To provide even this peculiar answer, which may be called a symbolic root of the equation, the *pair of numbers* 3 and 2 are needed as well as the i attached to the 2.

The 3 and 2 cannot be interchanged. They make an *ordered pair* of numbers. This combination $3+2i$ is called a *complex number* of which the 3 is called the *real* part and the $2i$ the *imaginary* part.

The next step in dealing with complex numbers was the discovery of a geometrical representation for them and that i can be regarded as an operator; but before proceeding to this, it is well to get used to working with the symbol i—which is often also written j—using the rule $i^2=-1$.

Note that $a+ib$ is only equal to $a'+ib'$ if $a=a'$ and $b=b'$. This is the *definition* of equality for two complex numbers.

If $a+ib=a'+ib'$, then $a-a'=i(b'-b)$.

Squaring gives $(a-a')^2=-(b'-b)^2$ which necessitates $a=a'$ and $b=b'$.

Note also that ia is the same as ai. It is usual however to write $3i$, and not $i3$.

Instances of addition and subtraction are

$$(3+5i)+(7-2i)=10+3i,$$
$$(4+2i)-(3+5i)=\ 1-3i.$$

For multiplication:

$$(4+6i)(5+3i)=20+30i+12i+18i^2=20-18+42i=\ 2+42i,$$
$$(6-5i)(2+3i)=12-10i+18i-15i^2-12+15+\ 8i=27+\ 8i,$$
$$(2+3i)(2-3i)=\ 4-\ 9i^2=13.$$

These last factors are called *conjugate* complex numbers, and their product is real; $(a-ib)$ is the conjugate of $(a+ib)$.

This helps in division; thus $\dfrac{20+5i}{4+3i}=\dfrac{(20+5i)(4-3i)}{(4+3i)(4-3i)}$.

The numerator $=80-15i^2+20i-60i=95-40i$.

The denominator $=16-9i^2=16+9=25$.

$\therefore$ the fraction $=\dfrac{95-40i}{25}=\dfrac{19-8i}{5}$.

Examples 98

1. Show that $(13+5i)^2=144+130i$ and find the product
$$(13+5i)(5+13i).$$

2. Show that the symbol i disappears from the products

 (i) $(5+3i)(5-3i)$, (ii) $(7+4i)(7-4i)$, (iii) $(a+ib)(a-ib)$,

 and find the value in each case.

3. Show that $(x-5-7i)(x-5+7i)=x^2-10x+74$, and state the roots of $x^2-10x+74=0$.

4. (*a*) Show that $(37+3i)\div(2-3i)=5+9i$:

 (i) by multiplying out $(2-3i)(5+9i)$;

 (ii) by multiplying the L.H.S. by $(2+3i)/(2+3i)$.

 (*b*) Show $\dfrac{a+ib}{c+id}$ as a complex number by multiplying it by $\dfrac{c-id}{c-id}$.

5. Show that $(5+6i)(3-2i)=27+8i$ and deduce factors for $27-8i$.

 Hence show that $(5^2+6^2)(3^2+2^2)=27^2+8^2$.

 Use a similar method to express $(4^2+7^2)(9^2+5^2)$ as the sum of two squares. Repeat for $10\times 34=(9+1)(25+9)$ and for 17×61.

6. (i) Work out $(a+ib)(c+id)$ and $(a+ib)(c-id)$.
(ii) Prove that $(1+i)^2+(1-i)^2=0$.

7. Express the powers of i up to i^8 in their simplest form.

8. (i) If $r=\sqrt{(a^2+b^2)}$ and $\tan\theta=b/a$ so that

$$\cos\theta : \sin\theta : 1 = a : b : \sqrt{(a^2+b^2)},$$

show that $a+ib=r(\cos\theta+i\sin\theta)$.
(ii) Work out $(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)$.

[As quadratics may be factorised by being expressed as the difference of two squares and using the factors of a^2-b^2, so the result $a^2+b^2=(a+ib)(a-ib)$ may be used to give complex factors, e.g.

$$\begin{aligned} a^2+6ab+20b^2 &= a^2+6ab+9b^2+11b^2 \\ &= (a+3b)^2+(b\sqrt{11})^2 \\ &= (a+3b+ib\sqrt{11})(a+3b-ib\sqrt{11}).] \end{aligned}$$

9. Factorise as products of complex numbers :

(i) x^2+7y^2; (ii) x^2+xy+y^2; (iii) a^2-ab+b^2;
(iv) $a^2+8ab+20b^2$; (v) $x^2-10x+30$; (vi) $7+x^2$.

10. Find the quadratic equation whose roots are $a+ib$ and $a-ib$.

11. If the roots of $x^2+6x+13=0$ are $-a\pm ib$, form the equation whose roots are $-b\pm ia$.

12. Prove that $(a^2+b^2)(c^2+d^2)(e^2+f^2)$ may be expressed as the sum of two squares. Indicate several different answers.

13. Expressing $\dfrac{3x^2-3x+8}{(x-3)(x^2+4)}$ as $\dfrac{A}{x-3}+\dfrac{B}{x-2i}+\dfrac{C}{x+2i}$,

find A, B, C by the cover-up rule; show also that the last two fractions add up to make $x/(x^2+4)$.

14. Express $\dfrac{1}{x(x^2+4)}$ as partial fractions, in the form

$$\frac{A}{x}+\frac{B}{x-2i}+\frac{C}{x+2i}$$

and hence, in the form

$$\frac{A}{x}+\frac{Dx+E}{x^2+4}.$$

15. Factorise $x^2-2x+10$ and express $\dfrac{5x^2-14x+8}{(x-2)(x^2-2x+10)}$ as the sum of three partial fractions, using the cover-up rule to find the numerators. Give also the real partial fractions.

Coefficients and Symmetric Functions of Roots

It was shown in Chapter V by a process which applies whether the numbers concerned are real or complex, that if $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$, then

$$\Sigma\alpha_1 = -\frac{a_1}{a_0},\ \Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0},\ \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \dots .$$

If the coefficients in the equation, viz. a_0, a_1, a_2, etc., are real, then $\Sigma\alpha_1$, $\Sigma\,\alpha_1\alpha_2$, etc., must be real, in spite of some or all of the "α"s being complex.

This is because, in an equation *with real coefficients*, complex roots must occur in pairs which are of the form $a+ib$ and $a-ib$.

These two complex numbers, $a \pm ib$, are called *conjugate* complex numbers.

Example I. The roots of an equation are $\alpha+i\beta$, $\alpha-i\beta$, γ, δ where α, β, γ, δ are real.

Show that the sums of the products of these roots (i) two at a time, (ii) three at a time are real.

(i) The products two at a time are:

$$(\alpha+i\beta)(\alpha-i\beta),\ (\alpha+i\beta)\gamma,\ (\alpha-i\beta)\gamma,\ (\alpha+i\beta)\,\delta,\ (\alpha-i\beta)\,\delta,\ \gamma\delta.$$

The sum of these is $\alpha^2+\beta^2+2\alpha\gamma+2\alpha\delta+\gamma\delta$, which is real.

(ii) The products three at a time are:

$$(\alpha+i\beta)(\alpha-i\beta)\gamma,\ (\alpha+i\beta)(\alpha-i\beta)\,\delta,\ (\alpha+i\beta)\gamma\delta,\ (\alpha-i\beta)\gamma\delta.$$

The sum of these is

$$(\alpha^2+\beta^2)\gamma+(\alpha^2+\beta^2)\,\delta+2\alpha\gamma\delta, \text{ which is real.}$$

Examples 99

1. Find by direct multiplication the sum of the products three at a time of

   (i) $2+3i,\ 2-3i,\ 5+i,\ 5-i$;

   (ii) $3+i,\ 3-i,\ 3+2i,\ 3-2i,\ 2$.

   Also form the equations with these roots and verify the results from the relations above.

2. Use the identity $\Sigma x^2 \equiv (\Sigma x)^2 - 2\Sigma x\beta$ to find the sum of the squares of the roots of the equations:

   (i) $x^3+3x^2+x+203=0$; (ii) $x^4+x^2+1=0$;

   (iii) $x^4-1=0$.

   Verify the answer to (iii) by finding the roots and squaring.

3. If the real root of $x^3+px+q=0$ is $x=-2a^2$ show that the other roots are of the form $a^2 \pm ib$ where $a^2(a^4+b^2)=q$. Deduce that the other roots are real if $a^6 > q^2$.

4. If α, β, γ are the roots of $x^3 - 3x^2 + 7x - 5 = 0$ write down the values of $\Sigma\alpha$, $\Sigma\alpha\beta$ and find the value of $\Sigma\alpha^2$.

   Use $\alpha^3 = 3\alpha^2 - 7\alpha + 5$ and similar relations to find the value of (i) $\Sigma\alpha^3$, (ii) $\Sigma\alpha^4$.

5. Show that the roots of the equation in Example 4 are 1, $1 + 2i$, $1 - 2i$, and evaluate directly (i) $\Sigma\frac{1}{\alpha}$; (ii) $\Sigma\left(\frac{1}{\alpha}\right)^2$; (iii) $\Sigma\alpha^5$.

6. Show that $x = a$ is a root of $x^3 + ax^2 - (a^2 + b^2)x - a(a^2 - b^2) = 0$.

   Find the sum and product of the other two roots and hence the quadratic equation which they satisfy.

7. Find the sum of (i) the fifth powers, (ii) the sixth powers of the roots of $x^5 + px - q = 0$.

Geometrical Representation of Complex Numbers; i as an Operator

Any real number, e.g. 2 or $\sqrt{3}$, may be represented by a *point* on a *line* (the x-axis) when the origin O has been chosen, or by the *displacement* from O to the point. If the number is negative the displacement is to the left instead of to the right.

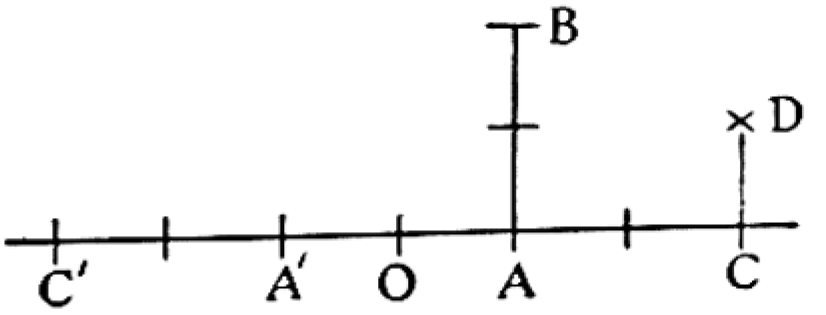

FIG. 44 (i)

If in Fig. 44 (i) OA represents a displacement a, then $3a$ is represented by OC; thus the 3 has operated on OA to stretch it to 3 times its length.

Again, $-1 \times OA$ is OA'; the -1 has operated on OA to reverse its direction; also $-3 \times OA$ is OC', the -1 reversing and the 3 stretching the original displacement.

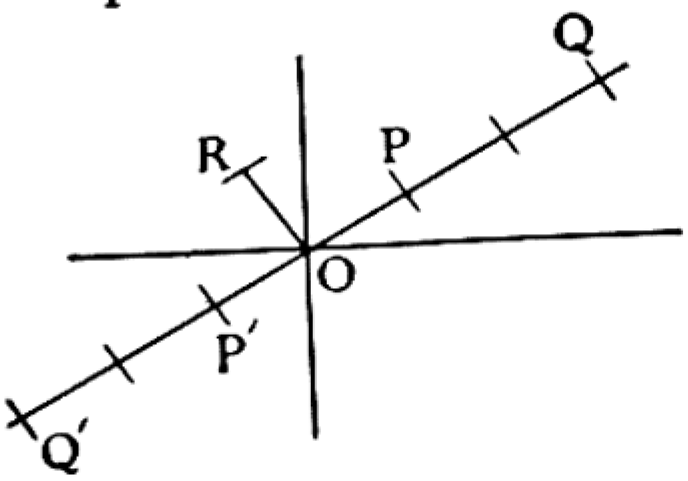

FIG. 44 (ii)

Any complex number may be represented by a point in the *plane* or by the *displacement* from the origin O to the point.

If in Fig. 44 (ii) the displacment is OP, then

$$3 \,.\, OP \text{ is } OQ, \quad -1 \,.\, OP \text{ is } OP', \quad -3 \,.\, OP \text{ is } OQ'.$$

Now suppose that, instead of thinking of OP' as OP reversed, we think of *OP' as OP rotated through 2 right angles* (in a counter-clockwise direction), and suppose we define j as the operator which rotates a displacement through 1 right angle (in a counter-clockwise direction), then $j \,.\, OP$ will be OR and $j \,.\, OR$ will be OP'.

But $j \,.\, OR = jj \,.\, OP$, which is naturally written $j^2 \,.\, OP$.

Thus $j^2 \,.\, OP = OP' = -1 \,.\, OP$.

In this sense $j^2 = -1$ or $j = \sqrt{(-1)}$.

In other words, this operator j is what we have been calling i.

[Both the letters i and j are commonly used in this sense, that is $i^2 = -1$ and $j^2 = -1$; j being more often used in Technical Colleges and i elsewhere. In this book, sometimes one will be used and sometimes the other.]

Again, in Fig. 44 (i) if $OA = 1$, then $AC = 2$, measuring the displacement from A instead of O, and $j \,.\, AC = AB$.

Thus $1 + 2j$ is represented by the displacement OB.

[$2j$ is the same as $j \,.\, 2$; it is usual to write ja but $2j$].

In Fig. 44 (i) OD represents $3 + j$ while AD represents $2 + j$, the displacements not necessarily being measured from the origin.

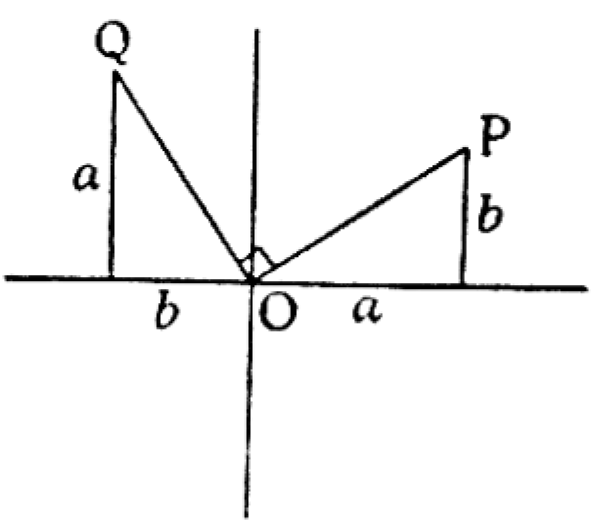

FIG. 45

In Fig. 45, $\angle QOP$ is a right angle. So $j \;\, OP = OQ$ since j rotates OP through a right angle. Also OP is $a + jb$ and

$$\begin{aligned} j(a + jb) &= ja - b \text{ or } -b + ja, \\ &= OQ \end{aligned}$$

so the usual multiplication rule gives the same result.

To sum up:

$a + ib$ (or $a + jb$) is represented geometrically by the point of the plane whose coordinates are a and b.

What has been called above the *displacement OP* is often called the *vector OP*. The disadvantage of the word *vector* is that, while the rules for the addition of vectors are the same as for the addition of complex numbers, the rules for multiplication are not the same in the two cases.*

The method of representing complex numbers by points in a

* See *Report* (*Math. Assoc.*) *on Teaching of Trigonometry in Schools*, Chapter XI.

plane is often called the *Argand diagram*, after Argand who first expounded the idea clearly.

The de Moivre form for a complex number

If in Fig. 46 $OP = r$ and $\angle NOP = \theta$, so that the polar coordinates of P are (r, θ), then $a = r\cos\theta$, $b = r\sin\theta$.

$$\therefore\ a + ib = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta).$$

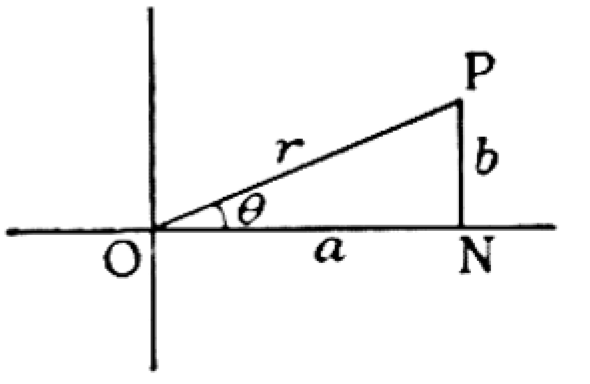

FIG. 46

This is called the de Moivre form of a complex number. It leads to what is known as de Moivre's theorem (p. 273).

r is called the *modulus* and θ the *argument* of the complex number.

In terms of a and b, the modulus, usually denoted by $|a + ib|$, is $\sqrt{(a^2 + b^2)}$.

Since $r\{\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)\} = r(\cos\theta + i\sin\theta)$ for integral values of n, the argument of the complex number $a + ib$ is many-valued; however, the value of θ in the range $-\pi < \theta \leqslant \pi$ is defined as the *principal value* of the argument, and is the one usually given. This can be seen to imply that the argument is (i) $\tan^{-1}\dfrac{b}{a}$ if $a > 0$, (ii) $\tan^{-1}\dfrac{b}{a} + \pi$ if $a < 0 < b$, (iii) $\tan^{-1}\dfrac{b}{a} - \pi$ if $a < 0$ and $b < 0$. (See Examples 100, No. 5.)

The word *amplitude* is often used instead of *argument* to denote the angle θ, but the latter is preferred as it avoids any confusion which might arise by using *amplitude* here and in dealing with vibrations.

The abbreviation used for the argument of $a + ib$ is $\arg(a + ib)$.

Multiplication of Complex Numbers in de Moivre Form

Note that $ir(\cos\theta + i\sin\theta) = r(i\cos\theta - \sin\theta)$.

Now $-\sin\theta = \cos\left(\theta + \dfrac{\pi}{2}\right)$ and $\cos\theta = \sin\left(\theta + \dfrac{\pi}{2}\right)$.

$$\therefore\ i \,.\, r(\cos\theta + i\sin\theta) = r\left\{\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right\},$$

which is in agreement with

multiplying by i turns a vector through a right angle.

Again, $r(\cos\theta+i\sin\theta)\times r'(\cos\phi+i\sin\phi)$ by direct multiplication

$$=rr'\{\cos\theta\cos\phi+i^2\sin\theta\sin\phi+i(\sin\theta\cos\phi+\cos\theta\sin\phi)\}.$$

Now $$\cos\theta\cos\phi-\sin\theta\sin\phi=\cos(\theta+\phi)$$
and $$\sin\theta\cos\phi+\cos\theta\sin\phi=\sin(\theta+\phi).$$
$\therefore$ the product is $rr'\{\cos(\theta+\phi)+i\sin(\theta+\phi)\}$.

Hence the rule:

To multiply two complex numbers, multiply the moduli and add the arguments. ....(i)

No greater or less for Complex Numbers

When two points on the x-axis such as A and C Fig. 44 (i) represent real numbers, we can say that the point further from O represents the larger of the two numbers.

But with points on the plane such as B and D we cannot say that one of the two numbers represented is greater than the other.

We could say that D represents the number with the greater *modulus*, while B represents the number with the greater *argument*, but the numbers as wholes cannot be compared in this way.

In particular we cannot say $a+ib>0$ or $a+ib<0$.

Examples 100

1. If $4+2i=r(\cos\theta+i\sin\theta)$ show that $r=2\sqrt{5}$ and $\tan\theta=\frac{1}{2}$, drawing a diagram.

2. Work out $(a+ib)(c+id)$ and from your result deduce rule (i).

3. From rule (i) give the value of $r(\cos\theta+i\sin\theta)\div\rho(\cos\alpha+i\sin\alpha)$.

   Verify that the result $\dfrac{20+5i}{4+3i}=\dfrac{19}{5}-\dfrac{8}{5}i$ fits with your answer.

4. If $z=x+iy$ and $z'=x-iy$, z' is the number said to be *conjugate* to z.

   (i) Show z and z' on one Argand diagram.

   (ii) Show by multiplication that

   $zz'=x^2+y^2$, $z^2+z'^2=2(x^2-y^2)$, $z^3+z'^3=2x(x^2-3y^2)$.

   (iii) If $z=r(\cos\theta+i\sin\theta)$ give the de Moivre form of z', and obtain the value of $z^3+z'^3$ by using these forms.

5. Show on a diagram $\arg(a+ib)$ and $\tan^{-1}\dfrac{b}{a}$ for the four cases when $a=\pm 3$, $b=\pm 4$, remembering that $\tan^{-1}\dfrac{b}{a}$ lies between $-\pi/2$ and $+\pi/2$. State the connection between the two angles in each case.

6. If in Fig. 47 A is the point (a, a'), B is (b, b') and $OBCA$ is a parallelogram, show that

(i) OC represents $(a+ia')+(b+ib')$, i.e. $\overline{OC} \equiv \overline{OA}+\overline{OB}$.

(ii) AB is equal and parallel to the vector representing

$$(b+ib')-(a+ia'),$$

i.e. $\overline{AB} \equiv \overline{OB}-\overline{OA}$.

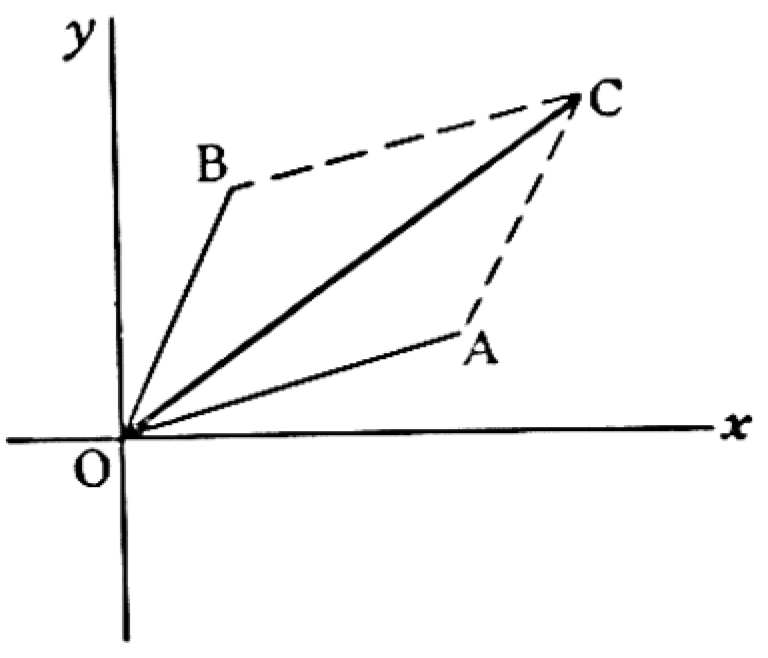

FIG. 47

In Nos. 7, 8, 9 cis θ is used as an abbreviation for cos $\theta + i$ sin θ.

7. Use the Argand diagram to show that

$$\operatorname{cis}\theta+\operatorname{cis}\left(\theta+\frac{2\pi}{3}\right)+\operatorname{cis}\left(\theta+\frac{4\pi}{3}\right)=0.$$

8. Show on an Argand diagram the points P, P', P'' representing respectively the complex numbers

$$r \operatorname{cis}\theta,\ r' \operatorname{cis}\theta',\ rr' \operatorname{cis}(\theta+\theta'),$$

and show that if A is the point $(1, 0)$, Δs AOP', POP'' are similar.

9. Show on an Argand diagram the points P, P', P'' representing respectively $a \operatorname{cis}\theta$, $\dfrac{1}{a}\operatorname{cis}\theta$, $\dfrac{1}{a \operatorname{cis}\theta}$.

Notice that $OP \,.\, OP' = a^2$, while P'' is the reflection of P' in the x-axis.

10. If $z \equiv x+iy$ and is represented by the point P in the Argand diagram, describe the locus of P if

(i) $|z|=1$; (ii) $|z-1|=2$; (iii) $|z-a-ib|=c$;

(iv) argument of $z=\frac{1}{3}\pi$; (v) argument of $(z-i)=\frac{1}{4}\pi$.

11. If $z_1 \equiv x_1+iy_1$ and $z_2 \equiv x_2+iy_2$ prove $|z_1+z_2| \leqslant |z_1|+|z_2|$.
What is the geometrical interpretation in the Argand diagram?
Also show $|z_1-z_2| \geqslant |z_1|-|z_2|$.

12. If P_1, P_2, P_3 represent the complex numbers z_1, z_2, z_3, show that the points are collinear if $z_3-z_1=k(z_2-z_1)$ where k is a real number.

13. If P represents the complex number z, give the relation z must satisfy if:

    (i) P lies inside or on the circle centre at (0, 0) and radius 2 units.

    (ii) P lies outside the circle centre at (1, 3) and radius 2 units.

    (iii) P lies on the line joining the origin to the point (1, 3).

The Notation of Ordered Pairs

In this notation the complex number $a+ib$ is written as $[a, b]$. The main laws obeyed by $a+ib$ are as follows.

(i) $a+ib=c+id$ if and only if $a=c$ and $b=d$.

(ii) $(a+ib)+(c+id)=(a+c)-i(b+d)$.

(iii) $(a+ib)-(c+id)=(a-c)+i(b-d)$.

(iv) $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$, using $i^2=-1$.

(v) $\dfrac{a+ib}{c+id}=\dfrac{(a+ib)(c-id)}{(c+id)(c-id)}=\dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$.

In the notation of ordered pairs, these laws read:

(i)′ $[a, b]=[c, d]$ if and only if $a=c$ and $b=d$.

(ii)′ $[a, b]+[c, d]=[a+c, b+d]$.

(iii)′ $[a, b]-[c, d]=[a-c, b-d]$.

(iv)′ $[a, b]\times[c, d]=[ac-bd, ad+bc]$.

(v)′ $\dfrac{[a, b]}{[c, d]}=\left[\dfrac{ac+bd}{c^2+d^2}, \dfrac{bc-ad}{c^2+d^2}\right]$.

Note also that the pair $[a, 0]$ behaves just as the real number a.

Now suppose that the 5 laws (i)′ to (v)′ are laid down as *defining* the relations between the *ordered pairs* of numbers, written as $[a, b]$. *There is now no need to introduce the "imaginary" $\surd-1$, or the operator i or j.*

We can show, to return to the first example, that $[3, 2]$ and $[3, -2]$ both satisfy the equation $x^2-6x+13=0$.*

For if x is $[3, 2]$, x^2 is $[3, 2]\times[3, 2]=[9-4, 6+6]=[5, 12]$

$-6x+13$ is $-6[3, 2]+13=[-18, -12]+13$,

and the sum of these is $[-13, 0]+13$,

which we regard as 0 taking $[1, 0]$ to behave exactly as 1.

Similarly $[3, -2]$ will be found to be such that x^2-6x comes to $[-13, 0]$.

* Strictly speaking, it is the equation $[1, 0]x^2-[6, 0]x+[13, 0]=0$ which is satisfied by $x=[3, \pm 2]$.

It is worth while to examine the *units* in the ordered pair notation. In the place of the usual 1 and -1 we have the four units

$$[1, 0],\ [0, 1],\ [-1, 0],\ [0, -1].$$

In addition and subtraction these units behave as would be expected, for we have $[1, 0]+[1, 0]=[2, 0]$ and $[0, 1]+[0, 1]=[0, 2]$; but in multiplication $[1, 0]\times[1, 0]$ gives $[1 . 1+0 . 0, 1 . 0+0 . 1]$, i.e. $[1, 0]$, while

$$[0, 1]\times[0, 1]=[0 . 0-1 . 1, 0 . 1+1 . 0]=[-1, 0], \text{(compare } i^2=-1);$$

also $\quad [1, 0]\times[0, 1]=[1 . 0-0 . 1, 1 . 1+0 . 0]=[0, 1].$

We conclude that the complex unit $[1, 0]$ behaves exactly as the unit in arithmetic; $[0, 1]$ behaves as we have supposed i to behave.

Also $[a, b]=[a, 0]+[0, b]=a[1, 0]+b[0, 1]$.

Examples 101

1. Use rule (v)′ to show that $\dfrac{[9, 12]}{[3, 4]}=[3, 0]$.
2. Use rule (iv)′ to find $[a, b]^2$.
3. Use the rules to work out the squares of the negative units $[-1, 0]$ and $[0, -1]$.
4. Work out $[a, b]^3+[a, -b]^3$.
5. Show that the usual factors for x^2-y^2 are still valid, if x and y are the ordered pairs $[a, b]$, $[c, d]$.
6. Express $\dfrac{[a, b]}{[a, -b]}$ in the form $[c, d]$.
7. Find real numbers x and y such that
$$x[5, 4]+y[4, 3]+1=0.$$
8. If $[a, b]^2=[-21, 20]$ find a, b.
9. Find the cube of $\left[-\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$.
10. Evaluate:
(i) $[a, b][a, b]$; (ii) $[a, -b][a, -b]$; (iii) $[a, b][a, \ b]$;
(iv) $[a, b]\div[a, b]$; (v) $[a, b]\div[a, -b]$.

De Moivre's Theorem

In the result

$$r(\cos\theta+i\sin\theta)\times r'(\cos\phi+i\sin\phi)=rr'\{\cos(\theta+\phi)+i\sin(\theta+\phi)\},$$

proved on p. 270, put r and r' each 1 and $\phi=\theta$.

We get $\quad (\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta.$

Again $(\cos 2\theta + i \sin 2\theta)(\cos\theta + i\sin\theta) = \cos 3\theta + i \sin 3\theta$,

that is $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$.

Proceeding in this way, we get the result that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

if n is a positive integer.

This is De Moivre's Theorem for a positive integral index.

When n is a negative integer $(= -m)$ we can proceed as follows:

$$\begin{aligned}(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} &= \frac{1}{(\cos\theta + i\sin\theta)^m} \\ &= \frac{1}{\cos m\theta + i\sin m\theta} \\ &= \cos m\theta - i\sin m\theta.\end{aligned}$$

Now $\cos m\theta = \cos(-m)\theta$ and $\sin m\theta = -\sin(-m)\theta$; hence again we have $(\cos\theta + i\sin n\theta)^n = \cos n\theta + i\sin n\theta$.

The case when n is fractional needs special care. Suppose $n = \frac{p}{q}$, where p and q are positive integers.

Then $(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \{(\cos\theta + i\sin\theta)^p\}^{\frac{1}{q}} = \{\cos p\theta + i\sin p\theta\}^{\frac{1}{q}}$.

It must be noticed at this stage that

$$\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = \cos p\theta + i\sin p\theta,$$

and also that

$$\begin{aligned}\left(\cos\frac{p\theta + 2\pi}{q} + i\sin\frac{p\theta + 2\pi}{q}\right)^q &= \cos(p\theta + 2\pi) + i\sin(\theta + 2\pi) \\ &= \cos p\theta + i\sin p\theta.\end{aligned}$$

Hence while we can say that $\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is *a* qth root of $\cos p\theta + i\sin p\theta$, it is not the only one, since $\cos\frac{p\theta + 2\pi}{q} + i\sin\frac{p\theta + 2\pi}{q}$ is another qth root.

Other roots are obtained by giving integral values to r in

$$\cos\frac{p\theta + 2r\pi}{q} + i\sin\frac{p\theta + 2r\pi}{q},$$

and r different roots are given by r taking the values $0, 1, 2, \ldots, (q-1)$.

Giving r the values $q, q+1, q+2, \ldots$ repeats these roots.

Examples 102

1. Expand $(\cos\theta + i\sin\theta)^n$ for $n=2, 3, 4$ and so obtain expressions for $\sin 2\theta$, $\cos 3\theta$, $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.
2. Writing 1 as $\cos 2r\pi + i\sin 2r\pi$, find the three cube roots of 1, and show that their sum is zero.
3. Find the three cube roots of -1 by considering
$$\cos(\pi + 2r\pi) + i\sin(\pi + 2r\pi).$$
4. If $z = \cos\theta + i\sin\theta$ and n a positive integer, show
$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$
By expanding $\left(z+\frac{1}{z}\right)^6$, express $\cos^6\theta$ in terms of $\cos 6\theta$, $\cos 4\theta$, $\cos 2\theta$.
5. If $z = \cos\theta + i\sin\theta$, find the values of $\left(z^5+\frac{1}{z^5}\right)$ and of $\left(z^5-\frac{1}{z^5}\right)$.
6. Find the four fourth roots of (-1) and show them on an Argand diagram.
7. Show that the fourth powers of $z \equiv \cos\pi/6 + i\sin\pi/6$ is equal to iz.
8. Use de Moivre's Theorem to simplify:

   (i) $\left(\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10}\right)^5$; (ii) $\dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$;

   (iii) $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta + i\sin 3\theta}$; (iv) $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta - i\sin 3\theta}$.
9. By first solving the equation $z^5 = 1$, show that
$$z^5 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right).$$
$$\left[\text{Hint.}\quad \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}.\right]$$
10. Show that
$$\frac{z^7-1}{z-1} = \left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right).$$

The Cube Roots of Unity

If $x^3 = 1$, then $x^3 - 1 = 0$;
$$\therefore\ (x-1)(x^2+x+1) = 0;$$
$$\therefore\ x = 1 \text{ or } x = \tfrac{1}{2}\{-1 + i\surd 3\} \text{ or } x = \tfrac{1}{2}\{-1 - i\surd 3\},$$
or in the de Moivre form $x = 1$ or $\cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}$.

These are the three cube roots of unity; a knowledge of their properties is needed for the general solution of cubic equations.

The square of $\frac{1}{2}\{-1+i\sqrt{3}\}$ is $\frac{1}{4}\{1-3-2i\sqrt{3}\}=\frac{1}{2}\{-1-i\sqrt{3}\}$.

If therefore the first of the complex cube roots is called (as is usual) ω, the other is equal to ω^2.

Again, the square of $\frac{1}{2}\{-1-i\sqrt{3}\}$ is $\frac{1}{4}\{1-3+2i\sqrt{3}\}=\frac{1}{2}\{-1+i\sqrt{3}\}$, so that the square of ω^2 is equal to ω.

This is also seen because $(\omega^2)^2=\omega^4=\omega^3 \,.\, \omega=\omega$, since $\omega^3=1$.

Thus the three cube roots are 1, ω, ω^2, and it is not necessary to specify which of the complex roots we call ω.

Also, since ω and ω^2 are the roots of $x^2+x+1=0$ we have

$$\omega^2+\omega+1=0$$

or

$$\omega^2+\omega=-1,$$

which can be seen by adding their worked-out values.

Examples 103

1. Prove that $(1+\omega)^2+(1-\omega)^2=-2\omega$
$$=\tfrac{1}{2}\{(1-\omega)^2+(1+\omega)^2\}.$$

2. Prove:
(i) $(1+\omega^2)^3=-1$; (ii) $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$;
(iii) $(\omega-1)^3=3(2\omega+1)$; (iv) $\dfrac{\omega}{(1-\omega)^2}=-\dfrac{1}{3}$.

3. If $5+7i=l\omega+m\omega^2$, show that $l=-5+\frac{7}{3}\sqrt{3}$ and $m=-5-\frac{7}{3}\sqrt{3}$.
Express $a+ib$ as $p\omega+q\omega^2$.

4. Express $5+7i$ and $a+ib$ each in the form $l+m\omega$.

5. Prove that $x^3+1=(x+1)(x+\omega)(x+\omega^2)$ by multiplication.
Also prove this by changing the sign of x in
$$x^3-1=(x-1)(x-\omega)(x-\omega^2).$$
[Thus the cube roots of $\quad 1$ are -1, $-\omega$, $-\omega^2$.]

6. Prove that $(a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)=a^3+b^3+c^3-3abc$.

7. Show that the equation $x^3-3abx-(a^3+b^3)=0$ has roots $a+b$, $\omega a+\omega^2 b$, $\omega^2 a+\omega b$.

8. Show that the value of $1+\omega^n+\omega^{2n}$ for an integral value of n is 3 or 0, according as n is, or is not, a multiple of 3.

9. Show the seven seventh roots of unity on an Argand diagram. If z is one of the roots other than 1 and $\alpha=z+z^2+z^4$ while
$\beta=z^3+z^5+z^6$, show that (i) $\alpha+\beta=-1$ and (ii) $\alpha\beta=2$.

10. Find the value of:

(i) $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$; (ii) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ \omega^2 & \omega & 1 \end{vmatrix}$.

11. Express $\dfrac{x^2}{x^3+1}$ as partial fractions $\dfrac{A}{x+1}+\dfrac{B}{x+\omega}+\dfrac{C}{x+\omega^2}$, using the cover-up method to find A, B, C.

Also combine the two fractions containing ω to give a fraction independent of ω.

12. Express $\dfrac{3}{1-x^3}$ as the sum of 3 partial fractions (using ω and ω^2). Expand these fractions as power series in x, when $|x|<1$, and deduce the value taken by $1+\omega^n+\omega^{2n}$ for different values of n; check your result by expanding the original fraction.

Exp $i\theta$

In what follows it will be necessary to distinguish between

$$e^x \ (\simeq 2{\cdot}7^x) \text{ as a power of the number } e$$

and the series $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dots$, which is denoted by $\exp x$.

Having established that $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$, it may be expected that $\cos\theta+i\sin\theta$ can be associated with a function obeying an index law; for if $\cos\theta+i\sin\theta=f(\theta)$ we have

$$\{f(\theta)\}^n=f(n\theta).$$

Already it has been shown that if θ is measured in radians,

$$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\dots,$$

$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\dots,$$

and so
$$\cos\theta+i\sin\theta=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}\dots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}\dots\right)$$

$$=1+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^6}{6!}\dots$$

$$+i\theta+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^5}{5!}+\frac{(i\theta)^7}{7!}+\dots$$

$$=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\dots,$$

which is the result of substituting $i\theta$ for x in $\exp x$.

$$\therefore\ \cos\theta+i\sin\theta=\exp(i\theta). \quad \dots\dots\dots\dots\dots\text{(I)}$$

Now for real values of x, the series exp x obeys the index law

$$(\exp x)^n = \exp(nx) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

for $\exp(x+y)=e^{x+y}=e^x \,.\, e^y = \exp x \,.\, \exp y$ and the relation (II) follows.

Also the proof of de Moivre's theorem together with relation (I) prove that

$$\exp(i\theta)\,.\,\exp(i\phi)=\exp\{i(\theta+\phi)\} \quad \text{and} \quad \{\exp(i\theta)\}^n=\exp(in\theta).$$

Thus we are now in a position to define $e^{i\theta}$, which is at present meaningless, by writing $e^{i\theta} \equiv \exp(i\theta)$...(the series).

We have shown that $e^{i\theta}$, as thus defined, obeys the index laws and in fact behaves just as the real power e^x behaves.

Notice that $e^{i(\theta+2\pi)}=e^{i\theta}\times e^{i2\pi}$, which does not conflict with the fact that

$$\cos(\theta+2\pi)+i\sin(\theta+2\pi)=\cos\theta+i\sin\theta$$

since $$e^{i2\pi}=\cos 2\pi + i\sin 2\pi = 1 + i\,.\,0 = 1.$$

Notice also that

$$\frac{1}{\cos\theta+i\sin\theta}=\frac{\cos\theta-i\sin\theta}{\cos^2\theta+\sin^2\theta}$$
$$=\cos\theta-i\sin\theta=\cos(-\theta)+i\sin(-\theta),$$

just as $\frac{1}{e^{i\theta}}$ would be expected to be equal to $e^{-i\theta}$, i.e. to $e^{i(-\theta)}$.

Exponential Values of sine and cosine

Since $\quad e^{i\theta}=\cos\theta+i\sin\theta$ and $e^{-i\theta}=\cos\theta-i\sin\theta$,

by addition and subtraction we get

$$e^{i\theta}+e^{-i\theta}=2\cos\theta \quad \text{and} \quad e^{i\theta}-e^{-i\theta}=2i\sin\theta,$$

so that $$\cos\theta=\tfrac{1}{2}(e^{i\theta}+e^{-i\theta}) \quad \text{and} \quad \sin\theta=\frac{1}{2i}(e^{i\theta}-e^{-i\theta}). \quad \ldots\ldots\text{(III)}$$

These are the exponential values of $\cos\theta$ and $\sin\theta$. They should be compared with the *definitions* of the hyperbolic functions

$$\cosh x=\tfrac{1}{2}(e^x+e^{-x}) \text{ and } \sinh x=\tfrac{1}{2}(e^x-e^{-x}).$$

Examples 104

1. Write as exponential functions:

(i) $\cos 2\theta$; (ii) $\sin 3\theta$; (iii) $\cos pt + i\sin pt$;
(iv) $A\cos pt + B\sin pt$.

2. Show $1 - e^{i\theta} = -2i \sin \frac{1}{2}\theta \,.\, e^{i\theta/2}$.

3. If $\qquad C = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$

and $\qquad S = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$,

show $C + iS = (1 - e^{i(n+1)\theta})/(1 - e^{i\theta}) = \frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta \,.\, \{ie^{-i\theta/2} - ie^{i(n+\frac{1}{2})\theta}\}$.

Hence deduce the values of C and S by first writing the last expression in the form $A + iB$.

4. Prove (i) $1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = 0$;

(ii) $\sum_{r=1}^{n-1} \sin\frac{2r\pi}{n} = 0$ if n is a positive integer.

5. Working from the exponential values given in (III) above, as if these were the definitions of $\cos\theta$ and $\sin\theta$, prove that

(i) $\cos^2\theta + \sin^2\theta = 1$; $\qquad$ (ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta$;

(iii) $\sin 2\theta = 2\sin\theta\cos\theta$; (iv) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Also from the definitions, find the corresponding properties of $\cosh x$ and $\sinh x$.

6. Verify that the expected result $\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$ for differentiating a power of e agrees with the result obtained by differentiating $\cos\theta + i\sin\theta$, and also with that obtained by differentiating the series $\exp(i\theta)$ term by term.

7. Verify that $\frac{d}{d\theta}\left(\frac{1}{\cos\theta + i\sin\theta}\right) = \frac{d}{d\theta}(\cos\theta - i\sin\theta) = \frac{d}{d\theta}(e^{-i\theta})$.

The logarithm of a complex number

If z is the complex number $x + iy$, x and y being real, the series $\exp z \equiv 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$ is a series each of whose terms is complex, and separating these terms into their real and imaginary parts we get $\exp z$ as $P + iQ$ where both P and Q are convergent series of real terms. (Examples 105, No. 4.)

But it is simpler to proceed thus:

$$\exp z = e^{x+iy} = e^x \,.\, e^{iy} = e^x\{\cos y + i\sin y\},$$

which shows that $P = e^x \cos y$ and $Q = e^x \sin y$.

Again, if $\qquad x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} = e^{\log r + i\theta}$,

we can say that $\qquad \log(x + iy) = \log r + i\theta$, and hence

log (complex) = log (modulus) + i (argument).

Here, however, the relation

$$e^{i\theta} = \cos\theta + i\sin\theta = \cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi) = e^{i(\theta + 2n\pi)}$$

must be taken into account.

The "argument" of $x+iy$ is not definite, but if θ is the value lying between $-\pi$ and $+\pi$, the general value is $\theta+2n\pi$.

This can be allowed for by giving a capital letter to Log z for the general value, so that $\text{Log } z=\log r+i(\theta+2n\pi)$

and keeping $$\log z=\log r+i\theta \qquad (-\pi<\theta\leqslant\pi).$$

$\log z$ is called the *principal value* of the many-valued function Log z.

In all this it must be remembered that the logarithms are to base e.

Another way of writing the above results is

$$\log (x+iy)=\tfrac{1}{2}\log (x^2+y^2)+i\tan^{-1}\frac{y}{x}$$

where to avoid ambiguity in the imaginary part we have to take the principal value of the argument, viz. the value between $-\pi$ and $+\pi$.

Examples 105 (Remember that all logarithms are to base e)

1. What are the numbers whose logarithms are:

   (i) $2+i\dfrac{\pi}{3}$; (ii) $2+1{\cdot}5i$; (iii) $1-i\pi$?

2. What are the logarithms of the numbers:

   (i) $4+3i$; (ii) $1+i\sqrt{3}$; (iii) $1-i$?

3. If z and z' are conjugate complex numbers:

   (i) What connection is there between $\log z$ and $\log z'$?

   (ii) If $z=3+2i$ find $\log(z^2+z'^2)$.

4. Taking $z=r(\cos\theta+i\sin\theta)$ separate e^z into real and imaginary parts, $P+iQ$ as above.

5. From the definitions of Log z_1 and $\log z_2$, show that:

   (i) one value of (Log z_1 + Log z_2) is one value of Log (z_1z_2);

   (ii) the value of $(\log z_1+\log z_2)$ is not necessarily the value of $\log(z_1z_2)$ by considering the special case when

   $$z_1=z_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}.$$

Some Notations

As the expression $r(\cos\theta+i\sin\theta)$ occurs so frequently, various abbreviations for it are in use.

One of these

$$r\,\text{cis}\,\theta \quad (\text{cis}=c\,.\,i\,.\,s \text{ or } \cos\,.\,i\,.\,\sin)$$

has already been mentioned: cis is pronounced as siss.

Another is (r, θ), which may confuse the point in polar coordinates with the complex number.

In the ordered pair notation the expression becomes

$$[r \cos \theta, r \sin \theta],$$

but this is scarcely an abbreviation.

Finally $e^{i\theta}$ is almost defined as $\cos \theta + i \sin \theta$, being defined as the series which is equivalent to this.

In the various abbreviations, de Moivre's theorem that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

becomes

$$(\text{cis } \theta)^n = \text{cis } (n\theta)$$

or

$$(1, \theta)^n = (1, n\theta)$$

or

$$[\cos \theta, \ \sin \theta]^n = [\cos n\theta, \sin n\theta]$$

or

$$(e^{i\theta})^n = e^{in\theta}.$$

Functions of a Complex Variable

In a certain sense the function $x^2 + 2iy^2$ is a function of $x + iy$, for if $x + iy$ is known and hence x and y are separately known, then $x^2 + 2iy^2$ is known.

But more than this is implied when we speak of a function of z, where $z \equiv x + iy \equiv r(\cos \theta + i \sin \theta)$; when we say $w \equiv u + iv$ is a function of z we mean that w can be found from z *taken as a whole* by an algebraic formula or process and not merely from the parts x and y of z taken separately.

For instance, if $w = z^2$, then $w = (x + iy)^2 = x^2 - y^2 + i2xy$, or alternatively

$$w = (r \cos \theta + ir \sin \theta)^2 = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

In this case if $w = u + iv$, then

$$u = x^2 - y^2 = r^2 \cos 2\theta \quad \text{and} \quad v = 2xy = r^2 \sin 2\theta.$$

As another instance, if $w = \frac{1}{z}$, then

$$w = \frac{1}{x + iy} \times \frac{x - iy}{x - iy} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2};$$

or alternatively

$$w = \frac{1}{r \cos \theta + ir \sin \theta} = \frac{1}{r} (\cos \theta - i \sin \theta).$$

Here $\quad u = \frac{x}{x^2 + y^2} = \frac{1}{r} \cos \theta$ and $v = -\frac{y}{x^2 + y^2} = -\frac{1}{r} \sin \theta.$

Examples 106

Express in the form $u+iv$ the functions of z in Nos. 1 to 6, giving in each case both the Cartesian form (in x and y) and the polar form (in r and θ).

1. z^3. 2. $(z+i)^2$. 3. $(a+ib)z$.

4. $\dfrac{1}{z^2}$. 5. $z^2+\dfrac{1}{z^2}$. 6. $z-\dfrac{1}{z}$.

If z' is the complex number conjugate to z, state which of the following, Nos. 7 to 10, are functions of z and which functions of z', giving the function in each case.

7. x^2-y^2-2ixy. 8. $x^3-3xy^2-i(y^3-3x^2y)$.

9. $\dfrac{x}{x^2+y^2}+i\dfrac{y}{x^2+y^2}$. 10. $\dfrac{(x^2+y^2)^2}{x^2-y^2-2ixy}$,

Alter one sign in each of the following so that the result may be a function of z and state the function.

11. $x^4+6x^2y^2+y^4+i(4x^3y-4xy^3)$.
12. $(x^2+y^2)/\sqrt{(x^2+h^2-2ixy)}$.

The z-plane and the w-plane

If $y=x^2$, the values of x can be shown by points ranging from $x=-\infty$ to $x=+\infty$ on the x-axis and the corresponding values of y shown by ordinates measured perpendicular to this line, and so the "curve of squares" is obtained.

In the case, for instance, of $w=z^2$, the whole of the z-plane (or x, y plane) is needed to show the possible values of z,* and corresponding to a point $P(x, y)$ of the z-plane there is a point $P'(u, v)$ of a second plane, called the w-plane, where $u=x^2-y^2$ and $v=2xy$.

It follows that if P describes some locus in the z-plane, then the corresponding point P' will describe a locus in the w-plane. For the relation $w=z^2$, if P describes $x^2-y^2=a$ in the z-plane, P' will describe the line $u=a$ in the w-plane; or again, if P describes $2xy=b$ in the z-plane, P' describes $v=b$ in the w-plane.

Continuing with the transformation relation $w=z^2$, taking the form $z=r(\cos\theta+i\sin\theta)$, it follows that $w=r^2(\cos 2\theta+i\sin 2\theta)$.

This shows that

(i) to points in the z-plane which are on the circle centre at the origin and radius a correspond points in the w-plane on a circle radius a^2 and centre at the origin;

* What has been called "the z-plane" may be called the "*complex-number-plane*" for z and the "w-plane" the "*complex-number-plane*" for w.

(ii) the vectorial angle (or argument) in the w-plane is double that in the z-plane.

Thus in Fig. 48, where the circles have radii 2 and 4 units, the points A, A'; B, B'; C, C'; D, D' are pairs of corresponding points.

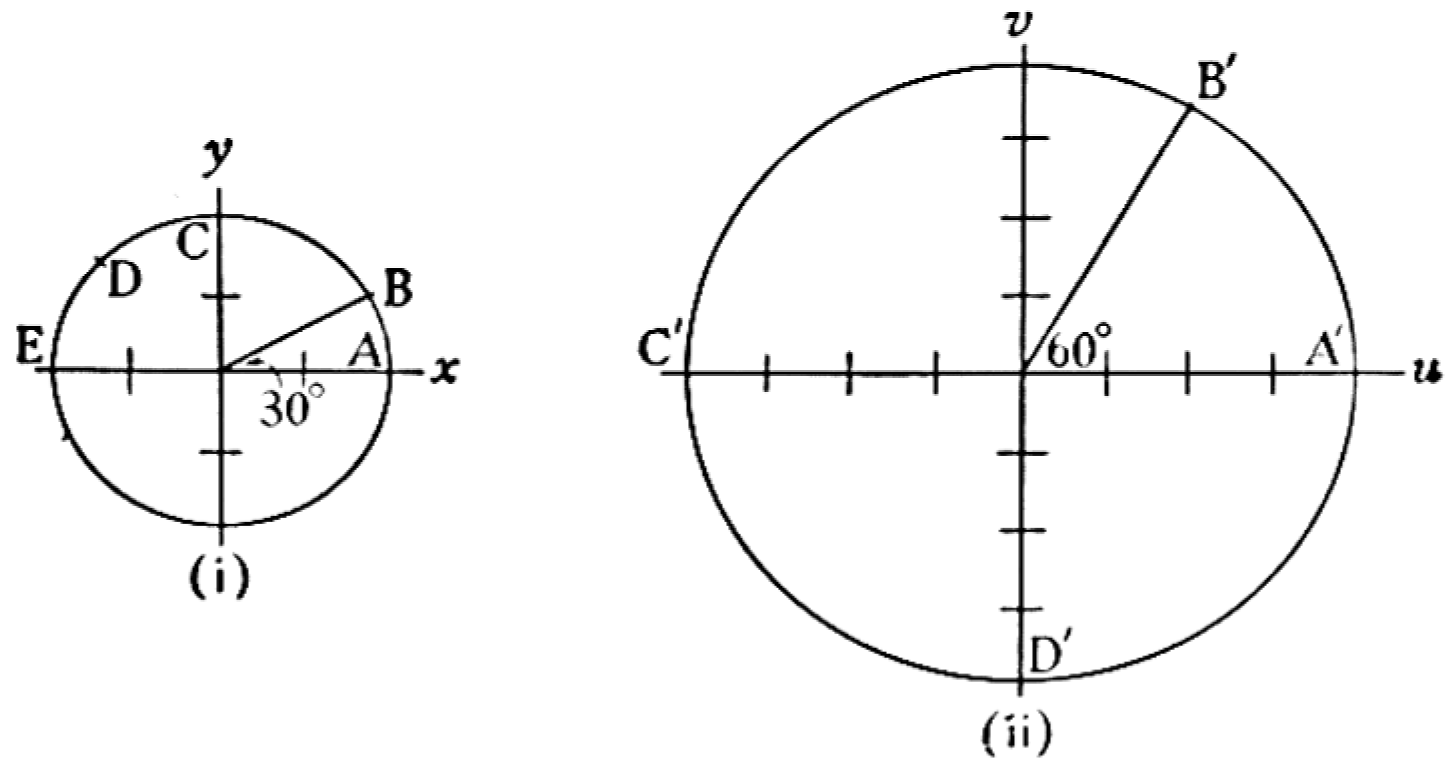

FIG. 48

Now suppose a point P in the z-plane to start at A and go round the circle in the anti-clockwise direction; the corresponding point P' in the w'-plane starts at A' and also goes round the circle in the anti-clockwise direction, but while P goes round its circle once, P' goes round its circle twice; in fact, if P ranges over the first two or upper quadrants of the z-plane, P' ranges over the whole of the w-plane.

Again, using the transformation equation $w=\frac{1}{z}$, where

$$z=r(\cos\theta+i\sin\theta) \quad \text{and} \quad w=\frac{1}{r}[\cos\theta+i\sin(-\theta)]$$

it follows that if P goes round the same circle in the z-plane (Fig. 48 (i)), then P' in the w-plane would go round a circle of radius $\frac{1}{2}$ in the opposite direction.

Examples 107

1. Discuss the transformation equations $w=z^3$ and $w=\frac{1}{z^2}$ in the way $w=z^2$ is discussed above.
2. Show that for the transformation $w=az+b$ where a and b are real numbers corresponding figures are similar.

   Consider the cases (i) $a=3, b=0$; (ii) $a=1, b=2$; (iii) $a=2, b=3$. Is it true if a and b are complex?

3. Show that $w=\frac{1}{z}$ gives a transformation which is the same as inversion in the unit circle centre at the origin followed by reflection in the x-axis.
4. If P describes the circle of unit radius centre at the origin in the z-plane and P' is the corresponding point in the w-plane, give the locus of P'

   (i) if $w=z-1$; (ii) if $w=\text{Log } z$.
5. Show that the transformation relation (ii) of Example 4 transforms points in the z-plane which lie between the two circles centre at the origin and radii a and b into points in the w-plane which lie between the lines $u=\log a$ and $u=\log b$.
6. If $P(x, y)$ in the z-plane is a point with positive x and lying between the lines $y=\frac{\pi}{6}$ and $y=\frac{\pi}{4}$, show that the transformation relation $w=e^z$ gives a corresponding point P' to the w-plane lying outside the unit circle centre at the origin and between two fixed lines through the origin. [Hint. Use $w=r'(\cos\theta'+i\sin\theta')=e^{x+iy}$.]
7. Show that the transformation relation $z=a\cosh w$, where $z\equiv x+iy$ and $w\equiv u+iv$, gives the relations

$$x=a\cosh u\cos v,\quad y=a\sinh u\sin v.$$

Hence show that the lines $u=$constant in the w-plane correspond to confocal ellipse in the z-plane.

Miscellaneous Examples 108

1. Prove that $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3=1$; what are the other numbers whose cube is 1?
2. If ω is a cube root of unity, prove that ω^2 is a second and that $1+\omega+\omega^2=0$. Also show ω^n is a root for all integral values of n.
3. Prove that $2ie^{i\frac{\pi}{6}}$ is a root of a quadratic equation with real coefficients, and find the other root.
4. If ω is a cube root of unity, other than 1, prove that

$$(a+b)(a+\omega b)(a+\omega^2 b)=a^3+b^3.$$

5. Give the modulus and the argument of the numbers:

   (i) $12+5j$; (ii) $1-j$; (iii) $\cos\frac{1}{2}\alpha-j\sin\frac{1}{2}\alpha$;
   (iv) $\sqrt{2}+\sqrt{3}+j(\sqrt{3}-\sqrt{2})$.
6. Simplify $(\text{cis }\theta)^2\div(\text{cis }-\theta)^2$.
7. Simplify $(\sin\theta-i\cos\theta)^3$.
8. What is the argument of

   (i) $1+\cos\theta+i\sin\theta$; (ii) $1-\cos\theta+i\sin\theta$?

9. What is the modulus of $(\sqrt{2}+i\sqrt{3})\div(\sqrt{3}+i\sqrt{5})$?

10. Simplify $\left(\dfrac{a+ib}{a-ib}\right)^2-\left(\dfrac{a-ib}{a+ib}\right)^2$.

11. If n is an integer of the form $3m\pm 1$ where m is also an integer, prove that $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^n+\left(\dfrac{-1-i\sqrt{3}}{2}\right)^n=-1$.

12. Factorise :

(i) x^5+1 in real factors ; (ii) x^6-1 in factors of first degree.

13. Prove :

(i) $1+2i+3i^2+4i^3+5i^4=\frac{1}{2}(6-4i)$;

(ii) $1+2i+3i^2+4i^3+\ldots$ to $(4m+1)$ terms $=2m+1-i2m$.

14. If n is an odd number greater than 3 but not a multiple of 3, show that a^2+ab+b^2 is a factor of $(a+b)^n-a^n-b^n$.

15. If 1, ω, ω^2 are the three cube roots of unity, show that

$$(y+\alpha+\beta)(y+\omega\alpha+\omega^2\beta)(y+\omega^2\alpha+\omega\beta)\equiv y^3-3\alpha\beta y+\alpha^3+\beta^3.$$

16. (i) If $a=p+q$, $b=\omega p+\omega^2 q$, $c=\omega^2 p+\omega q$, show $a+b+c=0$;

(ii) With the values for a, b, c given in (i) find the value of

$$a^2+b^2+c^2 \text{ and of } a^4+b^4+c^4.$$

Verify that if $a+b+c=0$, then $2(a^4+b^4+c^4)=(a^2+b^2+c^2)^2$.

CHAPTER XII

EXAMPLES OF INFINITE SERIES

Diagrams to illustrate convergence

Consider a series of positive terms. For such a series, S_n, the sum to n terms, either tends to a limit or increases indefinitely. The series can be represented on a diagram by marking the points (n, S_n) for values 1, 2, 3, ... of n as in Fig. 49.

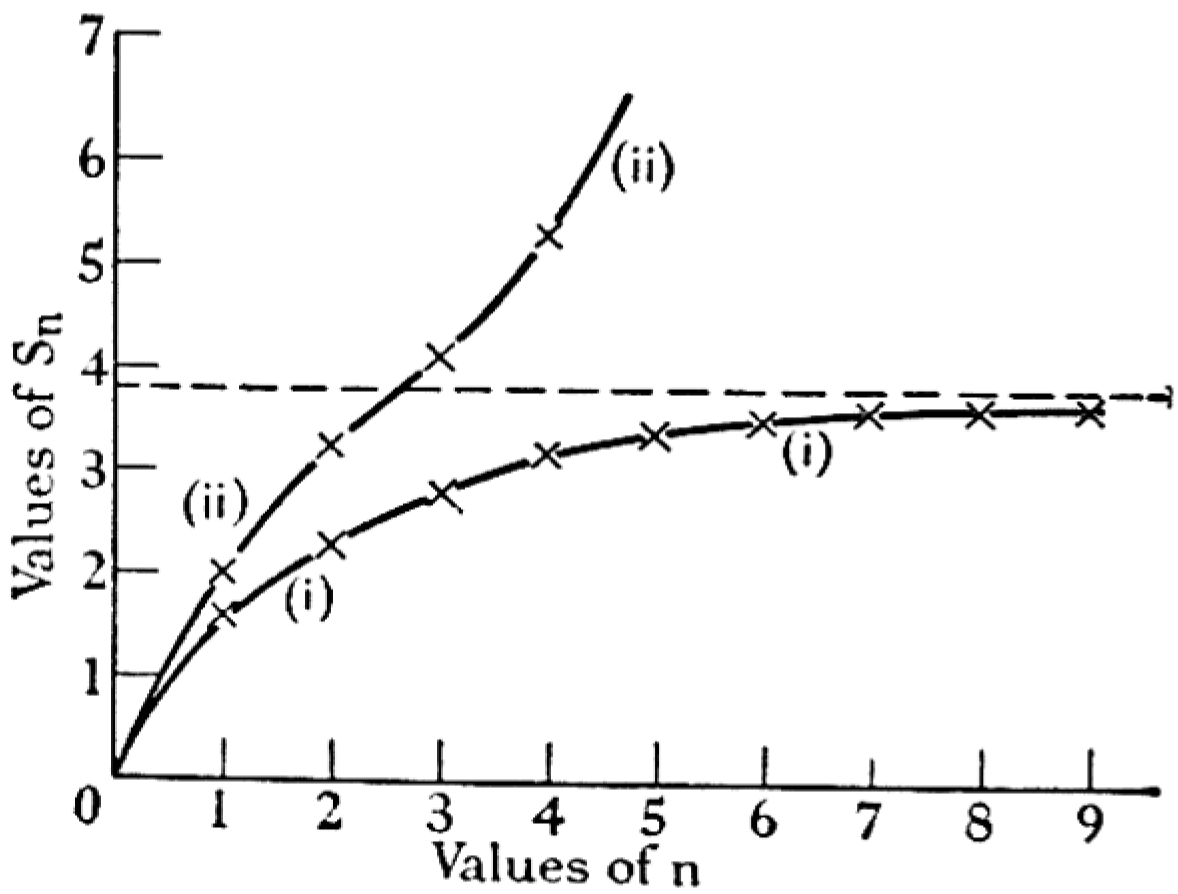

FIG. 49

A convergent series is represented by a curve such as (i) which rises to a horizontal asymptote, while a divergent series is represented by a curve such as (ii) which goes on rising indefinitely.

The same representation can, of course, be used if S_n is the nth term of a sequence and we are not considering a series, provided the terms of the sequence increase steadily.

Notice especially that for a series of positive terms there are only the two alternatives: either (i) the series converges to a finite limiting sum and the curve has a horizontal asymptote, or (ii) the series diverges and the curve rises to infinity.

The diagram makes clear a most important result.

If a series of positive terms is such that for all values of n, the sum $S_n <$ a constant k (independent of n) then it converges to a limit equal to or less than k; in other words, if in the diagram the curve is always

below the line $y=k$ it tends to a horizontal asymptote $y=p$ where $p=$ or $<k$.

In such a case the series is said to be "*bounded above*" and the result is stated by saying that "*a series of positive terms which is bounded above is convergent*".

The argument from the diagram should not be regarded as a strict proof, and the result should be regarded as a reasonable *assumption.*

It was a particular case of this assumption which was made in proving the exponential series convergent. Other examples of its use will now be given.

The reasoning given on p. 239 to show that the series for e^{-x} is convergent may be made much simpler and more convincing by the use of the idea of series "bounded above".

If a convergent series of positive terms is divided into two groups of terms, one called P and the other Q, then each of these groups is itself convergent, being bounded above by the sum $P+Q$ of the whole series. If then the signs of all the terms in the group Q are changed from $+$ to $-$, the resulting series will be convergent with sum $P-Q$.

This argument, which applies quite generally, can be applied in particular to the convergent series for e^x, from which that for e^{-x} is obtained by changing the sign of every second term.

Thus, the series for e^x may be written as

$$P+Q \equiv \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)+\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\right),$$

while that for e^{-x} may be written as

$$P-Q \equiv \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)-\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}\ldots+\right).$$

Example I. Convergence of $\Sigma\frac{1}{n^2}$. [Here and in Examples II to IV the Σ means $\sum\limits_{n=1}^{\infty}$.]

If we compare the series

$$\Sigma v_n \equiv \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{(n-1)n}+\frac{1}{n(n+1)}+\ldots$$

with $$\Sigma u_n \equiv \frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots+\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots$$

we see that each term of the second series is less than the corresponding term of the first series.

But the first series

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\ldots.$$

$\therefore$ its sum to n terms $=1-\dfrac{1}{n+1}$ which converges to the limit 1 as $n\to\infty$.

$\therefore$ the second series is convergent to a limit less than 1;
and, adding the usual first term, the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$$

converges to a limit less than 2.

Example II. Divergence of $\Sigma\dfrac{1}{n}$.

In contrast with the fact that $\Sigma\dfrac{1}{n^2}$ is convergent, it will now be proved that $\Sigma\dfrac{1}{n}$ is divergent.

Write $\quad 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots$
as $\quad 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\ldots.$

The first bracket is greater than $\frac{2}{4}$ or $\frac{1}{2}$.

The second bracket is greater than $\frac{4}{8}$ or $\frac{1}{2}$.

Similarly the next 8 terms are together greater than $\frac{8}{16}$ or $\frac{1}{2}$.

Thus taking groups of 2, 4, 8, 16, ... terms, we get in each case a total greater than $\frac{1}{2}$.

$\therefore$ the series $>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ to ∞, and S_n can be made greater than any named number.

Thus the series is divergent.

This shows that it is incorrect to suppose that a series must be convergent if the terms decrease and tend to zero as $n\to\infty$.

Example III. Show that $\Sigma\dfrac{1}{n^{5/2}}$ is convergent and $\Sigma\dfrac{1}{n^{1/2}}$ divergent.

$\dfrac{1}{n^{5/2}}<\dfrac{1}{n^2}$ and so $\Sigma\dfrac{1}{n^{5/2}}$ is "bounded above", being less than $\Sigma\dfrac{1}{n^2}$;

$$\therefore\ \Sigma\frac{1}{n^{5/2}}\text{ is convergent.}$$

$\dfrac{1}{n^{1/2}}>\dfrac{1}{n}$, and so $\Sigma\dfrac{1}{n^{1/2}}$ has its terms greater than the terms of a divergent series. $\therefore\ \Sigma\dfrac{1}{n^{1/2}}$ is divergent.

[The case of $\Sigma\dfrac{1}{n^p}$ when p lies between 1 and 2 cannot be dealt with in this way; but see Example 5, p. 290.]

Caution. With series of fractional terms such as we have been considering, care must be taken to see that no infinite terms occur.

Thus $\sum_{n=1}^{\infty} \frac{1}{(n-2)^2}$ is divergent, for the second term is infinite; but $\sum_{n=3}^{\infty} \frac{1}{(n-2)^2}$ is convergent, being $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Example IV. Prove that (i) $\Sigma \frac{1}{2n+1}$ is divergent; (ii) $\Sigma \frac{1}{(2n-5)^2}$ is convergent. For (ii) show that the sum is less than 5.

(i) $\frac{1}{2n+1} > \frac{1}{3n}$, but $\Sigma \frac{1}{3n} = \frac{1}{3} \Sigma \frac{1}{n}$ and is divergent.

$$\therefore\ \Sigma \frac{1}{2n+1} \text{ is divergent.}$$

(ii) $\frac{1}{(2n-5)^2} < \frac{1}{n^2}$, if $n^2 < (2n-5)^2$, which is true if $n > 5$;

$$\therefore\ \sum_{n=6}^{\infty} \frac{1}{(2n-5)^2} < \sum_{n=6}^{\infty} \frac{1}{n^2} \text{ and is convergent.}$$

The first 5 terms of the series are

$$\frac{1}{(-3)^2} + \frac{1}{(-1)^2} + \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \text{ which} = 2 + \frac{2}{9} + \frac{1}{25} < 3,$$

and $\Sigma \frac{1}{n^2}$ has been proved to converge to a sum less than 2.

$$\therefore\ \sum_{n=1}^{\infty} \frac{1}{(2n-5)^2} \text{ is certainly less than } 5.$$

Examples 109

1. For the series $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots$ find S_n and show that $S_n < \frac{1}{2}$.

 Find also the least value of n so that $\frac{1}{2} - S_n < \frac{1}{1000}$.

 Also deduce that the series $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots$ is convergent.

2. By finding the value of S_n show that the series

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots + \frac{1}{n(n+1)(n+2)} + \dots \text{ is convergent.}$$

 [Hint. the nth term $= \left\{\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right\} \div 2.$]

 Deduce that $\frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{(n+1)^3}$ is convergent.

3. Why does comparison with $\Sigma \frac{1}{n^2}$ show that $\Sigma \frac{1}{n^3}$ is convergent? To which sum will the first three terms give the better approximation?

4. Draw a diagram of curves through the points given by (n, S_n) for the series, marking 6 points in each case:

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$;
(ii) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$;
(iii) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \ldots$

[in (iii) joining the points by straight lines seems simplest as no special curve is suggested].

5. Treat $\Sigma \frac{1}{n^p}$ as $\Sigma \frac{1}{n}$ was treated in worked Example II, but start the brackets of 2, 4, 8, 16, ... terms one term earlier so that the series reads $1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots$.

Show that the series is less than $1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots$ and that this is a G.P. with common ratio $1/2^{p-1}$. Hence show that $\Sigma \frac{1}{n^p}$ is convergent if $p > 1$.

6. By comparing $\frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \ldots$ with $\frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} + \frac{1}{7 \cdot 8} + \ldots$, show that $\sum_{n=1}^{\infty} \frac{1}{(2n-5)^2}$, considered in Example IV, is less than 2·47.

7. State whether the series Σu_n is convergent or divergent if u_n is:

(i) $\frac{1}{n^2+1}$; (ii) $\frac{1}{2\sqrt{n}}$; (iii) $\frac{n}{(n+1)^2}$;

(iv) $\frac{5}{n^3}$; (v) $\frac{1}{n+\sqrt{n}}$; (vi) $\frac{1}{2^n \cdot n}$;

(vii) $\frac{1}{(2n+1)(2n+5)}$; (viii) $\frac{n-1}{n^3+1}$.

8. If $u_n = \log\left(1 + \frac{1}{n}\right)$ show that $u_n \to 0$ as $n \to \infty$ but that $\sum_{1}^{n} u_n = \log(n+1)$, so that Σu_n is divergent.*

Integrals and Series

The connection between $\Sigma f(n)\, dx$ and $\int f(x)\, dx$ can be shown by a diagram; an important special case is taken first, namely that of $\int \frac{1}{x}\, dx$ and $\Sigma \frac{1}{n}$.

* Due to P. H. Cody, *Math. Gazette*, Dec. 1955.

Since $\int_1^k \frac{dx}{x} = \log_e k$, the area above the x-axis and below the graph of $y = \frac{1}{x}$ from the point marked P to the point marked Q gives the value of $\log_e 5$.

Now look at the rectangles whose diagonals sloping upwards to the right are from 1 to A, 2 to B, 3 to C, 4 to D.

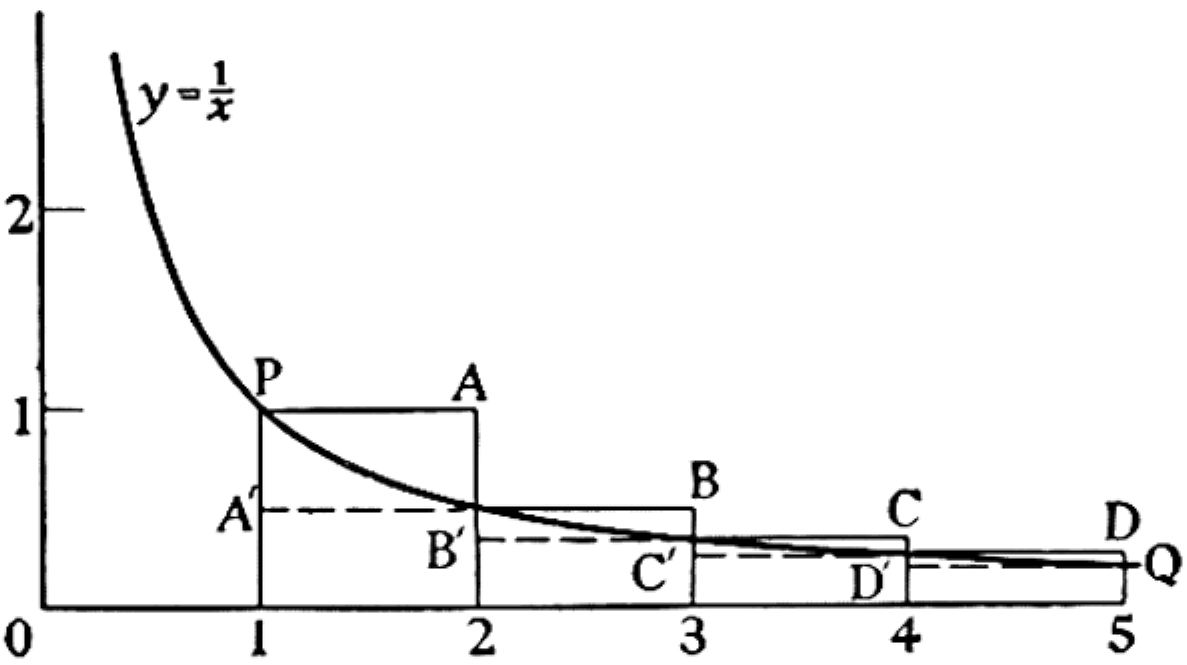

FIG. 50

These rectangles have a total area of $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, and this sum is greater than $\log_e 5$.

Again, if we look at the rectangles whose diagonals sloping upwards to the left are 2 to A', 3 to B', 4 to C', 5 to D'.

These rectangles have a total area of $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, and this sum is less than $\log_e 5$.

Thus the diagram shows that

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} < \log_e 5 < 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}.$$

The extension of this result is immediate:

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \dots + \frac{1}{n} < \log_e n < 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots + \frac{1}{n-1}. \qquad \dots\dots\dots\dots\dots(\alpha)$$

General Case

$\int f(x)\,dx$ when $f(x)$ is positive and *decreases* steadily as x increases from $x = 1$ can in the same way be compared with $\Sigma f(n)$.

In Fig. 51 the area above the x-axis and below the curve between the points A and B is given by $\int_1^8 f(x)\,dx$.

This is seen to be greater than the sum of the shaded rectangles from $x=1$ to $x=8$.

Thus $f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)<\int_1^8 f(x)\,dx.$

Also as in the special case of $\log x$, it can be seen that

$$f(1)+f(2)+f(3)+\ldots+f(7)>\int_1^8 f(x)\,dx.$$

The series thus provides limits between which the value of the integral lies.

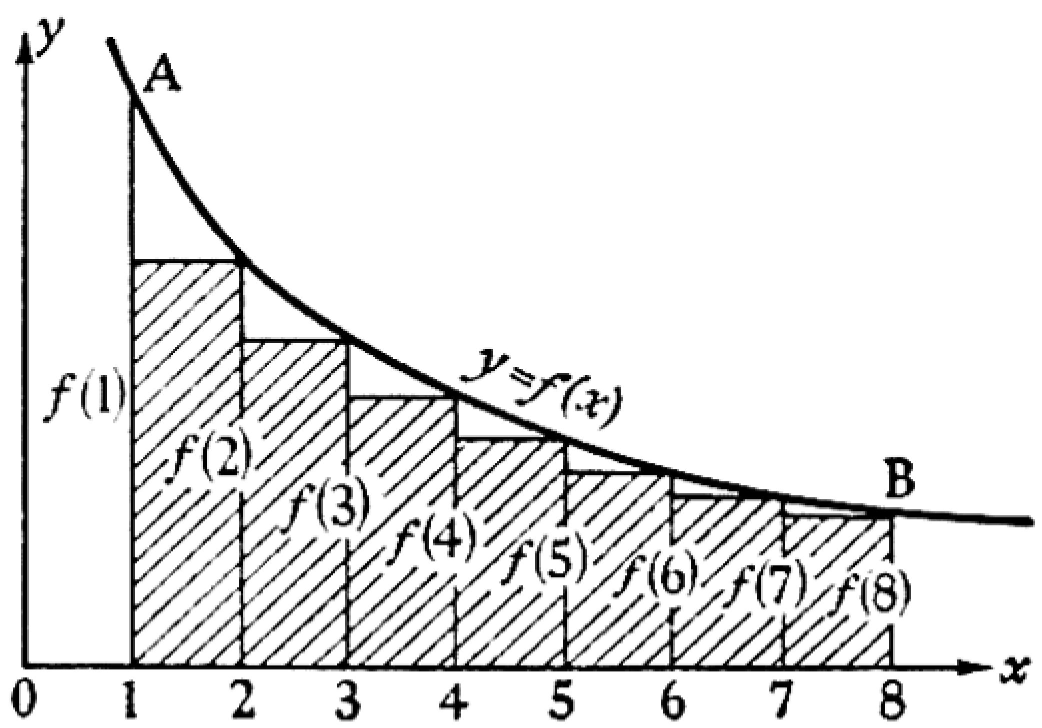

FIG. 51.

Again, considering only the rectangles below the curve, but supposing the steady decrease of $f(x)$ to continue as $x\to\infty$, the diagram shows that

if $\int_1^\infty f(x)\,dx$ is finite, then the series $\sum_{n=2}^{\infty} f(n)$ is convergent.

As an example of this, k being positive, it is known that

$$\int_1^\infty \frac{1}{x^{1+k}}\,dx=\left[-\frac{1}{k\,.\,x^k}\right]_1^\infty=+\frac{1}{k}.$$

$\therefore\ \sum_2^\infty \frac{1}{n^{1+k}}<\frac{1}{k}$, and so is a convergent series.

As the addition of $\frac{1}{1}$ makes no difference to the convergence, $\sum_1^\infty \frac{1}{n^{1+k}}$ is convergent if $k>0$ and converges to a limit $<1+\frac{1}{k}$.

[It will be noted that this is a shorter proof of the result than the one suggested in No. 5 of Examples 109.]

Examples 110

1. Use the results $\int_1^x \frac{dt}{t} = \log x$ and the curve $y = \frac{1}{t}$.

(*a*) to show that (i) $x > 1$, $\log x < x - 1$;
(ii) $x < 1$, $-\log x > 1 - x$.

[Thus $\log x < x - 1$ for any positive x.]

(*b*) to show that if t is positive $\log(1+t) > \frac{t}{1+t}$.

2. From $\frac{t}{1+t} < \log(1+t) < t$ if t is positive (proved in No. 1), deduce

(i) $\frac{1}{n+1} < \log\left(1+\frac{1}{n}\right) < \frac{1}{n}$; (ii) $\frac{n}{n+1} < \log\left(1+\frac{1}{n}\right)^n < 1$;

(iii) letting $n \to \infty$, $\operatorname*{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Also if x is positive and less than 1, then

(iv) $x < -\log(1-x) < \frac{x}{1-x}$;

(v) $1 - \frac{1}{x} < \log x < x - 1$;

(vi) $\log(x+y) - \log x > \frac{y}{x+y}$.

3. In No. 2 (ii) put $1 - \frac{1}{x}$ equal in succession to $\frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, \ldots$, prove that if m and n are integers with $n > m$, then

$$\log\frac{n}{m-1} > \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n}. \qquad \text{.................(i)}$$

Also putting $x - 1$ equal in succession to $\frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, \ldots$, prove that the sum on the R.H.S. of (i) is greater than $\log\frac{n+1}{m}$.

4. By comparing $\sum_{n=100}^{n=1000} \frac{1}{n^2}$ with $\int_{100}^{1000} \frac{1}{x^2}\,dx$, show that the Σ lies between ·009001 and ·0091.

[Remember it is Σ from 101 to 1000 which < the integral.]

5. Deduce from result (α), p. 291, that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{10} > \log_e 10 \text{ but } < \log_e 10 + 1,$$

and show that $\sum_{n=1}^{1000} \frac{1}{n}$ lies between 6·9 and 7·9.

Show also that the number of terms required to make the sum greater than 20 lies between 10^8 and 10^9.

6. Show by considering the area under $y=\frac{1}{x^3}$ that

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\dots+\frac{1}{n^3}<\frac{3}{2}-\frac{1}{2n^2}, \text{ and that } \sum_1^{\infty}\frac{1}{n^3}<\frac{3}{2}.$$

7. Show that (i) $\sum_1^{\infty}\frac{1}{1+n^2}<\frac{1}{2}+\frac{1}{4}\pi$;

(ii) $\sum_1^{\infty}\frac{1}{n^{3/2}}<3$.

Some Binomial Series

In the series for $(1-x)^{-m}$ every term is positive if $m>0$.

$$\text{For } (1-x)^{-m}=1+\frac{(-m)(-x)}{1}+\frac{(-m)(-m-1)}{1\,.\,2}(-x)^2$$
$$+\frac{(-m)(-m-1)(-m-2)}{1\,.\,2\,.\,3}(-x)^3+\dots$$
$$=1+\frac{m}{1}x+\frac{m(m+1)}{1\,.\,2}x^2+\frac{m(m+1)(m+2)}{1\,.\,2\,.\,3}x^3+\dots,$$

where x must be less than 1 in order that the series may be convergent.

If m is a fraction, series are obtained which it may not be easy to identify as binomial series.

Example I. Sum the series $1+\frac{4}{10}+\frac{4}{10}\cdot\frac{9}{20}+\frac{4}{10}\cdot\frac{9}{20}\cdot\frac{14}{30}+\dots$.

The 4, 9, 14 with differences 5, suggests $\frac{4}{5}(\frac{4}{5}+1)(\frac{4}{5}+2)$, and the term $\frac{4\cdot 9\cdot 14}{10\,.\,20\,.\,30}$ can be written $\frac{\frac{4}{5}(\frac{4}{5}+1)(\frac{4}{5}+2)}{1\,.\,2\,.\,3}\,.\,(\frac{1}{2})^3$, the earlier terms being $1+\frac{4}{5}\cdot\frac{1}{2}+\frac{\frac{4}{5}(\frac{4}{5}+1)}{1\,.\,2}\,.\,(\frac{1}{2})^2$.

Thus the series is seen to be the expansion of $(1-\frac{1}{2})^{-\frac{4}{5}}$, and the required value $=(\frac{1}{2})^{-\frac{4}{5}}=2^{\frac{4}{5}}\simeq$ antilog $\cdot 2408=1\cdot 741$.

Example II. Expand $(1-x)^{\frac{2}{3}}$ as far as the term in x^4, given $|x|<1$. Deduce the sum of the series

$$\frac{1}{6}x+\frac{1\cdot 4}{6\,.\,9}x^2+\frac{1\cdot 4\cdot 7}{6\,.\,9\,.\,12}x^3+\dots,$$

and hence show that $\frac{1}{12}+\frac{1\cdot 4}{12\,.\,18}+\frac{1\cdot 4\cdot 7}{12\,.\,18\,.\,24}+\dots=2-3(\frac{1}{2})^{\frac{2}{3}}$.

$$(1-x)^{\frac{2}{3}} = 1 - \tfrac{2}{3}x + \frac{\frac{2}{3}(-\frac{1}{3})}{1 \cdot 2}x^2 - \frac{(\frac{2}{3})(-\frac{1}{3})(-\frac{4}{3})}{1 \cdot 2 \cdot 3}x^3 + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{1 \cdot 2 \cdot 3 \cdot 4}x^4 - \dots$$

$$= 1 - \tfrac{2}{3}x - \tfrac{2}{3}x \cdot \tfrac{1}{6}x - \tfrac{2}{3}x \cdot \tfrac{1}{6}x \cdot \tfrac{4}{9}x - \tfrac{2}{3}x \cdot \tfrac{1}{6}x \cdot \tfrac{4}{9}x \cdot \tfrac{7}{12}x - \dots$$

$$= 1 - \frac{2}{3}x - \frac{2}{3}x\left\{\frac{1}{6}x + \frac{1 \cdot 4}{6 \cdot 9}x^2 + \frac{1 \cdot 4 \cdot 7}{6 \cdot 9 \cdot 12}x^3 + \dots\right\}.$$

$$\therefore \left\{(1-x)^{\frac{2}{3}} - 1 + \frac{2}{3}x\right\}\frac{3}{2x} = -\left\{\frac{1}{6}x + \frac{1 \cdot 4}{6 \cdot 9}x^2 + \frac{1 \cdot 4 \cdot 7}{6 \cdot 9 \cdot 12}x^3 + \dots\right\}.$$

$\therefore$ the first series to be summed $= \dfrac{3}{2x}\left\{1 - \dfrac{2}{3}x - (1-x)^{\frac{2}{3}}\right\}$.

The second series to be summed is obtained by putting $x = \frac{1}{2}$ in this result, so that the sum required is $3\{1 - \frac{1}{3} - (\frac{1}{2})^{\frac{2}{3}}\} = 2 - 3(\frac{1}{2})^{\frac{2}{3}}$.

Examples 111

1. Write down the first 5 terms in the expansion of $(1+x)^{-\frac{1}{2}}$ and deduce that the sum of the series

$$1 - \frac{1}{6} + \frac{1 \cdot 3}{6 \cdot 12} - \frac{1 \cdot 3 \cdot 5}{6 \cdot 12 \cdot 18} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{6 \cdot 12 \cdot 18 \cdot 24} - \dots \text{ is } \tfrac{1}{2}\sqrt{3}.$$

2. Write down the first 5 terms in the expansion of $(1-x)^{-\frac{2}{3}}$.

Deduce that the sum of the series $\dfrac{5}{12} + \dfrac{5 \cdot 8}{12 \cdot 18} + \dfrac{5 \cdot 8 \cdot 11}{12 \cdot 18 \cdot 24} + \dots$

is $3 \cdot 4^{\frac{1}{3}} - 4$.

3. Write down the first 6 terms in the expansion of $(1-x)^{\frac{3}{2}}$ and show that all terms after the second are positive $(0 < x < 1)$.

Deduce the sum of the series:

(i) $1 + \dfrac{1}{6}x + \dfrac{1 \cdot 3}{6 \cdot 8}x^2 + \dfrac{1 \cdot 3 \cdot 5}{6 \cdot 8 \cdot 10}x^3 + \dots$;

(ii) $1 + \dfrac{1}{12} + \dfrac{1 \cdot 3}{12 \cdot 16} + \dfrac{1 \cdot 3 \cdot 5}{12 \cdot 16 \cdot 20} + \dots$.

4. Use the expansion of $(1+x)^{\frac{1}{4}}$ to find the sum of the series

$$1 - \frac{7}{12}x + \frac{7 \cdot 11}{12 \cdot 16}x^2 - \frac{7 \cdot 11 \cdot 15}{12 \cdot 16 \cdot 20}x^3 + \dots.$$

5. Show that $1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$ are the first 4 terms of a binomial series, and find the coefficient of x^n in it.

6. Write down the coefficient of x^n in $(1-x)^{-\frac{1}{3}}$ and deduce the sum of the series $1 + \dfrac{1}{6} + \dfrac{1 \cdot 4}{2!\,6^2} + \dfrac{1 \cdot 4 \cdot 7}{3!\,6^3} + \dots$.

Series allied to the Standard Series

Series in which the term in x^n of a known series is multiplied by an integral function of n can often be summed as in the following examples.

Example I. Find the sum of the infinite series whose nth term is

$$\frac{(n+1)^2}{n!}.$$

$$u_n = \frac{(n+1)^2}{n!} \equiv \frac{n(n-1)+3n+1}{n!} = \frac{1}{(n-2)!} + \frac{3}{(n-1)!} + \frac{1}{n!}.$$

Having got the nth term in this form, it is advisable to look at the early terms of the series to see how they conform.

Now $u_2 = \frac{3^2}{2!} = \frac{1}{0!} + \frac{3}{1!} + \frac{1}{2!} = 1 + \frac{3}{1!} + \frac{1}{2!}$ (0! replaced by 1 fits),

$$u_1 = \frac{2^2}{1!} = 3 + \frac{1}{1!};$$

$$\therefore \sum_1^\infty \frac{(n+1)^2}{n!}$$

$$= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots\right) + 3\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots\right) + \left(\frac{1}{1!} + \frac{1}{2!} + \dots\right)$$

$$= \quad e \quad + \quad 3e \quad + \quad (e-1)$$

$$= 5e - 1.$$

Example II. Sum the series $\sum\limits_{n=1}^\infty (n^3 - n)\frac{x^n}{n!}$.

$$n^3 - n = n(n-1)(n-2) + 3n(n-1);$$

$$\therefore (n^3 - n)\frac{x^n}{n!} = \frac{x^n}{(n-3)!} + \frac{3x^n}{(n-2)!} = x^3 \cdot \frac{x^{n-3}}{(n-3)!} + 3x^2 \frac{x^{n-2}}{(n-2)!}.$$

This suggests two exponential series, one multiplied by x^3 and the other by $3x^2$.

Also the early terms are:

$$0x + (2^3 - 2)\frac{x^2}{2!} + (3^3 - 3)\frac{x^3}{3!} + (4^3 - 4)\frac{x^4}{4!}$$

$$= 3x^2 + 4x^3 + \frac{5x^4}{2}$$

$$= 3x^2 + 4x^3 + x^3\left(\frac{x}{1}\right) + 3x^2 \cdot \frac{x^2}{2!}$$

$$= 3x^2 + \left(x^3 \cdot 1 + 3x^2 \cdot \frac{x}{1}\right) + x^3\left(\frac{x}{1}\right) + 3x^2 \cdot \frac{x^2}{2!}.$$

Thus each of the two suggested exponential series is complete and the result is $3x^2\left(1 + \frac{x}{1} + \frac{x^2}{2!} + \dots\right) + x^3\left(1 + \frac{x}{1} + \frac{x^2}{2!} + \dots\right)$.

So the required sum is $(3x^2 + x^3)e^x$.

Example III. Find $\sum\limits_{1}^{\infty} \frac{n+1}{n} x^n$, given $|x| < 1$.

$$\text{The series} = \sum_{1}^{\infty} x^n + \sum_{1}^{\infty} \frac{x^n}{n}$$

$$= (x + x^2 + x^3 + \ldots) + \left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)$$

$$= \frac{x}{1-x} - \log(1-x).$$

Example IV. Find $\sum\limits_{n=1}^{\infty} (n^2 + 3n + 5) \frac{m(m-1)\ldots(m-n+1)}{n!} x^n$.

Write $n^2 + 3n + 5$ as $n(n-1) + 4n + 5$ and divide the term in x^n into 3 parts:

$$\frac{m(m-1)\ldots(m-n+1)}{(n-2)!} x^n,$$

$$4\frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n$$

and $$5\frac{m(m-1)\ldots(m-n+1)}{n!} x^n.$$

Of these three terms the first is

$$m(m-1)x^2 \cdot \frac{(m-2)(m-3)\ldots(m-n+1)}{(n-2)!} x^{n-2},$$

the second is $$4mx \frac{(m-1)(m-2)\ldots(m-n+1)}{(m-1)!} x^{n-1};$$

and, except for doubt as to the first few terms (which must be examined separately), these two when summed give

$$m(m-1)x^2 \cdot (1+x)^{m-2} \quad \text{and} \quad 4mx \cdot (1+x)^{m-1},$$

while the third term gives merely $5(1+x)^m$.

Now examine the early terms.

These are $$\frac{9mx}{1} + \frac{15m(m-1)}{2!} x^2 + \ldots$$

or $$5mx + 5\frac{m(m-1)}{2!} x^2$$

$$+ 4mx\{1 + (m-1)x + \ldots\}$$

$$+ m(m-1)x^2 \cdot 1.$$

Thus all the three series are complete except for the $5 \cdot 1$ of $5(1+x)^m$, and the sum is

$$m(m-1)x^2 \cdot (1+x)^{m-2} + 4mx(1+x)^{m-1} + 5(1+x)^m - 5.$$

Example V. Expand $\log (3+x)$ in ascending powers.

Given $\log 2 = \cdot 6931$ and $\log 3 = 1\cdot 0986$ calculate $\log 13$ (all these logarithms being to base e):

$$\log (3+x) = \log 3 + \log \left(1+\frac{x}{3}\right)$$

$$= \log 3 + \frac{x}{3} - \frac{1}{2}\frac{x^2}{9} + \frac{1}{3}\frac{x^3}{27} - \frac{1}{4}\cdot\frac{x^4}{81} + \dots .$$

In this take $x = \frac{1}{4}$, then

$$\log \tfrac{13}{4} = \log 3 + \log \left(1 + \tfrac{1}{12}\right);$$

$$\therefore \log 13 = 2 \log 2 + \log 3 + \frac{1}{12} - \frac{1}{2}\cdot\frac{1}{12^2} + \frac{1}{3}\cdot\frac{1}{12^3} - \frac{1}{4}\cdot\frac{1}{12^4}.$$

The work at the side shows that this

$$= 2\log 2 + \log 3$$
$$+ \cdot 08333 - \cdot 00347$$
$$+ \cdot 00019 - \cdot 00001$$

$$= 1\cdot 3862$$
$$+ 1\cdot 0986 - \cdot 0035$$
$$+ \quad \cdot 0835$$

$$= 2\cdot 5648$$

$$\begin{array}{r|l} 4 & \cdot 08333 \\ \hline 6 & \cdot 02083 \\ \hline 3 & \cdot 00347 \\ \hline 6 & \cdot 00116 \\ \hline 4 & \cdot 00019 \\ \hline 4 & \cdot 00005 \\ \hline & \cdot 00001 \end{array}$$

Examples 112

1. Find $\sum_1^\infty \frac{n^2}{n!}$.

2. Find $\sum_1^\infty (n^2+1)\frac{x^n}{n!}$.

3. Find $\sum_1^\infty \frac{(n^3+4)x^n}{n!}$.

4. Find $\sum_1^\infty (n+2)\frac{x^n}{n}$ if $|x|<1$.

5. Find $\sum_1^\infty (n^2+1)\frac{m(m-1)\dots(m-n+1)}{n!}x^n$.

6. Find $\sum_1^\infty (n^3+4)\frac{m(m-1)\dots(m-n+1)}{n!}x^n$.

7. Prove that $\sum_1^\infty (n+1)^2\frac{x^n}{n}$ if $|x|<1$

$$= \frac{x}{(1-x)^2} + \frac{2x}{1-x} - \log(1-x).$$

8. Express $(n+1)^3$ in the form $n(n-1)(n-2)+an(n-1)+bn+c$ and find $\sum_1^\infty (n+1)^3\frac{x^n}{n!}$.

9. Show that the coefficients of the given terms of the series

$$6x + \frac{6x^2}{2!} + \frac{7x^3}{3!} + \frac{11x^4}{4!} + \frac{29x^5}{5!} + \dots$$

are values of $[(n-1)!+5]/n!$ with $n = 1, 2, 3, 4, 5$.

Taking this function as determining the general coefficient find the sum of the series.

10. Show that $\sum_1^\infty \frac{1}{(n+2)\,.\,n!} = \frac{1}{2}$.

11. Expand $\log(2+x)$ in ascending powers.

Given $\log 2 = \cdot 6931$, calculate $\log 17$ to 4 figures (put $x = \frac{1}{8}$).

Series with Terms alternately + and −

Various important series, for example those for e^{-x} and for $\log(1+x)$, have their signs alternately + and −.

In both cases mentioned, if all the negative terms were changed to be positive, the new series would be convergent, as they would be the series for e^x, always convergent, and that for $-\log(1-x)$, convergent if $|x| < 1$.

This is, however, not always the case.

The series $$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots \quad \text{(i)}$$

will be proved convergent;

but the series $$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \dots \quad \text{(ii)}$$

has been proved divergent.

The series (i) can be written $(1-\frac{1}{2}) + (\frac{1}{3}-\frac{1}{4}) + (\frac{1}{5}-\frac{1}{6}) + \dots$, which shows that it is a positive quantity, each bracket being positive, and it also can be written

$$1 - (\tfrac{1}{2}-\tfrac{1}{3}) - (\tfrac{1}{4}-\tfrac{1}{5}) - (\tfrac{1}{6}-\tfrac{1}{7})\dots,$$

which shows that it is less than 1, so it cannot diverge.

Also it cannot oscillate, because the terms diminish steadily to zero. Therefore it is convergent.

This series is taken as typical of a "*conditionally convergent*" series, whereas the series for $\log(1+x)$ with $|x| < 1$, and the series for e^{-x} are called "*absolutely convergent*".

The series (i) is obtained from that for $\log(1+x)$ by putting $x = 1$ so it would be expected to be equal to $\log 2$. It actually is so, but this has not been proved; what happens at the *end* of the range of convergency is a difficult matter. At the other end of the range, if x is put equal to -1 in the series for $\log(1+x)$, we get the divergent series (ii) with all signs changed to minus.

The argument given to prove that the series (i) is convergent applies in the general case of a series $u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \ldots$ whose terms steadily diminish, provided $u_n \to 0$ as $n \to \infty$; for writing the series as $(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots$ we see that it is positive; and writing it as $u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ldots$ we see that it is less than u_1; also $u_n \to 0$ bars out oscillation.

Conditionally Convergent Series

Most of the frequently used infinite series are absolutely convergent.

Consideration of conditionally convergent series, and in particular of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

brings out the need for caution in reasoning about infinite series, and also that the "sum" of an infinite series is different from an ordinary sum of a finite number of terms.

We proceed to compare the "sum" of the series (i) with the "sum" of the series

$$(1 + \tfrac{1}{3} - \tfrac{1}{2}) + (\tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4}) + (\tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6}) + (\tfrac{1}{13} + \tfrac{1}{15} - \tfrac{1}{8}) + \ldots, \quad \ldots\text{(ii)}$$

which is obtained by rearranging the terms of (i).

Here the nth bracket is

$$\begin{aligned} &\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \\ &= \frac{8n^2 - 2n + 8n^2 - 6n - (16n^2 - 16n + 3)}{2n(4n-1)(4n-3)} \\ &= \frac{8n-3}{2n(4n-1)(4n-3)}. \end{aligned}$$

This is positive and the nth term of a convergent series since it behaves like $\dfrac{1}{4n^2}$ when n is large. Thus the series (ii) is convergent.

Now the sum of the series (i) $<$ the sum of the first three terms (since $-(\tfrac{1}{4} - \tfrac{1}{5}) - (\tfrac{1}{6} - \tfrac{1}{7}) - \ldots$ is negative), i.e. $< \tfrac{5}{6}$.

But all the brackets in series (ii) are positive, and so the sum of series (ii) $>$ the first bracket, i.e. $> \tfrac{5}{6}$.

Thus the "sums" of the series (i) and (ii), which appear to be made up of the same terms, are in fact different.

The student should note that no alteration in the order of the terms can alter the "sum" of an absolutely convergent series, though no proof of this statement is given in this book.

9. Show that the coefficients of the given terms of the series

$$6x+\frac{6x^2}{2!}+\frac{7x^3}{3!}+\frac{11x^4}{4!}+\frac{29x^5}{5!}+\dots$$

are values of $[(n-1)!+5]/n!$ with $n=1, 2, 3, 4, 5$.

Taking this function as determining the general coefficient find the sum of the series.

10. Show that $\sum_1^\infty \frac{1}{(n+2)\,.\,n!}=\frac{1}{2}$.

11. Expand $\log(2+x)$ in ascending powers.

Given $\log 2 = \cdot 6931$, calculate $\log 17$ to 4 figures (put $x=\frac{1}{8}$).

Series with Terms alternately + and −

Various important series, for example those for e^{-x} and for $\log(1+x)$, have their signs alternately $+$ and $-$.

In both cases mentioned, if all the negative terms were changed to be positive, the new series would be convergent, as they would be the series for e^x, always convergent, and that for $-\log(1-x)$, convergent if $|x|<1$.

This is, however, not always the case.

The series $\qquad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$ ....(i)

will be proved convergent;

but the series $\qquad 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots$ ....(ii)

has been proved divergent.

The series (i) can be written $(1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+\dots$, which shows that it is a positive quantity, each bracket being positive, and it also can be written

$$1-(\tfrac{1}{2}-\tfrac{1}{3})-(\tfrac{1}{4}-\tfrac{1}{5})-(\tfrac{1}{6}-\tfrac{1}{7})\dots,$$

which shows that it is less than 1, so it cannot diverge.

Also it cannot oscillate, because the terms diminish steadily to zero. Therefore it is convergent.

This series is taken as typical of a "*conditionally convergent*" series, whereas the series for $\log(1+x)$ with $|x|<1$, and the series for e^{-x} are called "*absolutely convergent*".

The series (i) is obtained from that for $\log(1+x)$ by putting $x=1$ so it would be expected to be equal to $\log 2$. It actually is so, but this has not been proved; what happens at the *end* of the range of convergency is a difficult matter. At the other end of the range, if x is put equal to -1 in the series for $\log(1+x)$, we get the divergent series (ii) with all signs changed to minus.

Examples 114

1. Show that $\operatorname*{Lt}_{x\to 0} (10^x - 1)/x \simeq 2{\cdot}3026$ and find $\operatorname*{Lt}_{x\to 0} (10^x - 5^x)/x$.

2. Find the limits as $n\to\infty$ of

$$\text{(i)}\ \left(1+\frac{x}{n}\right)^n;\quad \text{(ii)}\ \left(1+\frac{1}{n}\right)^{nx};\quad \text{(iii)}\ \left(1-\frac{x}{n}\right)^{nx};\quad \text{(iv)}\ \left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n.$$

3. If x is positive and n a large positive integer, so that $\frac{x}{n}$ is small,

prove that $\left(1+\frac{x}{n}\right)^n < \left(1+\frac{x}{n+1}\right)^{n+1}$.

4. Find the limits, x being positive :

$$\text{(i)}\ \operatorname*{Lt}_{x\to 0}\left\{\frac{\log(1+x)}{x}\right\};\qquad \text{(ii)}\ \operatorname*{Lt}_{x\to 0}\left\{\frac{e^x - 1 - \log(1+x)}{3x^2}\right\}.$$

5. (i) By division in ascending powers find the first four terms in the expansion of $\frac{1}{\log(1+x)}$.

(ii) By putting $x = 1 + y$ find $\operatorname*{Lt}_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\log x}\right)$.

6. What is the objection to the following reasoning?

$$\text{If } y = e^x,\ \frac{dy}{dx} = \operatorname*{Lt}_{h\to 0}\frac{e^{x+h}-e^x}{h} = \operatorname*{Lt}_{h\to 0} e^x\,.\,\frac{e^h - 1}{h}$$

$$= \operatorname*{Lt}_{h\to 0} e^x\left(1 + h + \frac{h^2}{2!} + \ldots - 1\right)\Big/ h = e^x.$$

Some Complex Series

In Chapter XI the important complex series

$$\exp(iy) \equiv 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \ldots$$

was discussed and proved equal to $\cos y + i\sin y$.

It was shown to be convergent, being equal to the sum of two convergent series, that for $\cos y$ and i times that for $\sin y$.

This series is a special type of complex series since the terms are alternately real and imaginary.

The more general complex series has real and imaginary parts in each term, being of the type

$$(u_1 + iv_1) + (u_2 + iv_2) + \ldots + (u_n + iv_n) + \ldots,$$

where the u's and v's are real.

This, however, is also the sum of two series:

$$(u_1+u_2+u_3+\ldots+u_n+\ldots) \text{ and } i(v_1+v_2+v_3+\ldots+v_n+\ldots),$$

and, like the special case, is convergent if each of the two series Σu_n and Σv_n is convergent.

It is not necessary as a rule to consider these series separately, for u_n and v_n are each less than $\sqrt{(u_n^2+v_n^2)}$, which is the *modulus* of the nth term of the complex series, and both the u-series and the v-series will be convergent if the series of the moduli, $\Sigma\sqrt{(u_n^2+v_n^2)}$ is convergent.

This is a *sufficient* but not a *necessary* condition; if it is satisfied, the complex series is said to be *absolutely convergent*.

For example, $\Sigma a^n (\cos n\theta + i \sin n\theta)$ is convergent, if Σa^n is convergent, that is if $|a|<1$, for

$$|\cos n\theta + i\sin n\theta| = \sqrt{(\cos^2 n\theta + \sin^2 n\theta)} = 1.$$

Power Series in $z \equiv x+iy$

Example I. $\exp z \equiv 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \ldots$ is always convergent.

It has been shown, p. 279, to be $\exp(x+iy) = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$.

Example II. The G.P. $1+z+z^2+z^3+\ldots$ is convergent if $|z|<1$. This can be summed to n terms or to infinity.

The sum to infinity $=\dfrac{1}{1-z}=\dfrac{1}{1-x-iy}=\dfrac{1-x+iy}{(1-x)^2+y^2}$.

Example III. $\sum\limits_{n=0}^{\infty} \cos^n\theta\,(\cos n\theta + i\sin n\theta)$.

The term set down is the $(n+1)$th term.

Its modulus $=|\cos\theta|^n \cdot |\cos n\theta + i \sin n\theta| = |\cos\theta|^n$.

$\therefore$ the series of moduli is convergent, so that the given series is convergent, if $|\cos\theta|<1$.

This is true for all values of θ except $\theta=n\pi$. If $\theta=n\pi$, $\sin n\theta=0$ and $\cos^n\theta \cdot \cos n\theta$ is either $(+1)(+1)$ if n is even or $(-1)(-1)$ if n is odd; so that in either case the series is $1+1+1+\ldots$ and is divergent.

Examples 115

1. Show that $e^{x+in\pi}$ is real.

2. Show that $a+ib=r\exp i\alpha$ when $r=\sqrt{(a^2+b^2)}$ and $\tan\alpha=\dfrac{b}{a}$.

   If $b=a\sqrt{3}$ prove the condition for absolute convergence for the series $1+(a+ib)+(a+ib)^2+(a+ib)^3+\ldots$ is $|a|<\frac{1}{2}$.

   Find the sum of the series to 9 terms.

3. Discuss the convergence of the series $\sum_{n=0}^{\infty} \sin^n \theta (\cos n\theta + i \sin n\theta)$ as in worked Example III.

4. Sum the series $1 + i\pi + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} + \dots$ to ∞.

5. Discuss the convergence of the series $\sum_{n=0}^{\infty} x^n (\cos n\theta + i \sin n\theta)$ and find its sum when convergent by using $\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n$ and summing the G.P.

Hence write down the sum of the two series

$$\text{(i)} \sum_{n=0}^{\infty} x^n \cos n\theta; \qquad \text{(ii)} \sum_{n=1}^{\infty} x^n \sin n\theta.$$

6. Show if $|x| < 1$ that the series

$$\cos\theta + x\cos 3\theta + x^2 \cos 5\theta + \dots \text{ (i) is convergent.}$$

Also by summing the series

$(\cos\theta + i \sin\theta) + x(\cos\theta + i\sin\theta)^3 + x^2(\cos\theta + i \sin\theta)^5 + \dots$, of which (i) is the real part,

show the sum of (i) to be $(1-x)\cos\theta/(1 - 2x\cos 2\theta + x^2)$.

7. Sum the series, given $|x| < 1$:

(i) $1 + 2x\cos\theta + 3x^2 \cos 2\theta + 4x^3 \cos 3\theta + \dots$;

(ii) $2x \sin\theta + 3x^2 \sin 2\theta + 4x^3 \sin 3\theta \dots$.

Comparison with the G.P.

In comparing a series Σu_n with a standard series Σv_n known to be convergent two methods have been used.

When Σu_n was $\Sigma \frac{1}{n^2}$, the series $\Sigma v_n = \Sigma \frac{1}{(n-1)n}$ was shown to be convergent, and since $\frac{1}{n^2} < \frac{1}{(n-1)n}$, for $n > 1$, $\Sigma \frac{1}{n^2}$ was convergent also.

But in dealing with the exponential series, the *ratio* of each term to the previous one was found and it was shown that after a certain term this ratio was always less than $\frac{1}{2}$; thus by comparison with a G.P. of ratio $\frac{1}{2}$, the exponential series was shown to be convergent.

Other examples of this method of comparing ratios will now be given.

Example I. Discuss the convergence of the series

$$\frac{2}{4}x + \frac{2 \cdot 4}{4 \cdot 7}x^2 + \frac{2 \cdot 4 \cdot 6}{4 \cdot 7 \cdot 10}x^3 + \frac{2 \cdot 4 \cdot 6 \cdot 8}{4 \cdot 7 \cdot 10 \cdot 13}x^4 + \dots \text{ (x positive).}$$

The nth term is $\frac{2 \cdot 4 \cdot 6 \dots 2n}{4 \cdot 7 \cdot 10 \dots (3n+1)} x^n$.

The ratio of this to the previous term is $\dfrac{2n}{3n+1}x$.

This ratio $<\frac{2}{3}x$ and so the given series is convergent if $\frac{2}{3}x<1$, for the terms will decrease more rapidly than those of the G.P. whose ratio is $\frac{2}{3}x$.

$$\tfrac{2}{3}x<1 \quad \text{if} \quad x<\tfrac{3}{2}.$$

So the series is convergent for all values of $x<\frac{3}{2}$.

Caution. If $x=\frac{3}{2}$ the ratio of the nth term to the previous one is $\dfrac{6n}{6n+2}$ and this is actually less than 1; but it gets nearer and nearer to 1 as n increases, and as $n\to\infty$ the ratio $\to 1$.

This state of affairs *does not make the series convergent.* Compare the fact that for the divergent series $\Sigma\dfrac{1}{n}$ the ratio of the nth term to the previous one is $\dfrac{n-1}{n}$, which is less than 1 but tends to 1 as $n\to\infty$.

Note. If x were allowed to take negative values, the series would be convergent if $|\frac{2}{3}x|<1$, i.e. if $|x|<\frac{3}{2}$.

Thus the series is convergent if $-\frac{3}{2}<x<\frac{3}{2}$.

Example II. Show that (x positive) the series $\dfrac{x}{1\cdot 2}+\dfrac{x^2}{2\cdot 3}+\dfrac{x^3}{3\cdot 4}+\dots$ is convergent if $x<1$.

Find the sum of the series.

[It is usual to take the ratio of the $(n+1)$th term to the nth, rather than that of the nth to the $(n-1)$th.]

For the series $u_n=\dfrac{x^n}{n(n+1)}$, $\qquad u_{n+1}=\dfrac{x^{n+1}}{(n+1)(n+2)}$;

$\therefore\ \dfrac{u_{n+1}}{u_n}=\dfrac{n}{n+2}\cdot x$, which $\to x$ as $n\to\infty$.

$\therefore$ if $x<1$ the terms will become less than those of a convergent G.P. [for example, if $x=\cdot 9$, the terms will become less than those of a G.P. for which the common ratio is $\cdot 95$.]

$\therefore$ the series is convergent.

Note. It is also convergent if x is negative and $|x|<1$.

To find the sum, write the series as

$$\frac{x}{1}-\frac{x}{2}+\frac{x^2}{2}-\frac{x^2}{3}+\frac{x^3}{3}-\frac{x^3}{4}+\frac{x^4}{4}-\frac{x^4}{5}+\dots.$$

The positive terms are $\dfrac{x}{1}+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\dfrac{x^4}{4}+\dots=-\log(1-x)$.

The negative terms are $-\frac{x}{2}-\frac{x^2}{3}-\frac{x^3}{4}-\frac{x^4}{5}-\dots$

$$= -\frac{1}{x}\left\{\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\dots\right\}$$

$$= -\frac{1}{x}\{-\log(1-x)-x\}.$$

$\therefore$ the whole sum $= 1+\frac{1}{x}\log(1-x)-\log(1-x)$.

Ratio Test

The method used for the exponential series on p. 238 and in these examples can be stated as a general test for convergence for a series Σu_n.

If $\frac{u_{n+1}}{u_n} < k$ for all sufficiently large n and $|k| < 1$, the series is convergent. This is called d'Alembert's Test.

Examples 116

1. For what ranges of values of x (being positive) is the series

$$\frac{2}{3}+\frac{2\cdot 4}{3\cdot 8}x+\frac{2\cdot 4\cdot 6}{3\cdot 8\cdot 13}x^2+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 8\cdot 13\cdot 18}x^3+$$

convergent?

2. Show that $1+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+\dots$ is convergent if $|x|<1$ and also if $x=-1$.

3. Show that $\frac{2}{1^2}+\frac{2^2}{2^2}+\frac{2^3}{3^2}+\frac{2^4}{4^2}+\dots+\frac{2^n}{n^2}+\dots$ is divergent.

4. Show that $\Sigma\frac{n^2}{(n+1)!}$ is convergent.

5. Considering $a<1$, $a=1$ and $a>1$ separately, show that, if a is positive, the series

$$\frac{1}{1+a}+\frac{a}{1+a^2}+\frac{a^2}{1+a^4}+\dots+\frac{a^n}{1+a^{2n}}+\dots$$

is always convergent except when $a=1$.

Miscellaneous Examples 117

Show that the following infinite series are convergent (Nos. 1 to 5) if $0<x<1$.

1. $\cdot\dot{8}\dot{9}$.

2. $(1+1^2)x+(1+2^2)x^2+(1+3^2)x^3+\dots$.

3. $\frac{2}{3}+\frac{2\cdot 3}{3\cdot 5}+\frac{2\cdot 3\cdot 4}{3\cdot 5\cdot 7}+\dots+\frac{2\cdot 3\cdot 4\dots n}{3\cdot 5\cdot 7\dots(2n-1)}+\dots$.

4. $1+\frac{1!\,1!}{2!}x+\frac{2!\,2!}{4!}x^2+\frac{3!\,3!}{6!}x^3+\dots$.

5. $\frac{1^2}{2!}+\frac{2^2}{3!}+\frac{3^2}{4!}+\dots+\frac{n^2}{(n+1)!}+\dots$.

6. Show that $\Sigma\frac{3^n}{n^3}$ is divergent.

7. Find the sum of the series $1-\frac{2}{3}+\frac{3}{3^2}-\frac{4}{3^3}+\frac{5}{3^4}-\dots$ by comparing it with the expansion of $(1+x)^{-2}$.

8. Find the sum of the series $1-\frac{1}{2^2}+\frac{1\cdot3}{2!}\cdot\frac{1}{2^4}-\frac{1\cdot3\cdot5}{3!}\cdot\frac{1}{2^6}+\frac{1\cdot3\cdot5\cdot7}{4!}\cdot\frac{1}{2^8}-\dots$ by comparing it with the expansion of $(1+x)^{-\frac{1}{2}}$.

9. Apply d'Alembert's test for convergence to the series whose nth term is $n^{\frac{1}{2}}/n!$ to show it convergent.

Show also that the series is convergent by comparing it with the infinite series for e.

10. Show that the series $\frac{2}{1^2}+\frac{3}{2^2}x+\frac{4}{3^2}x^2+\frac{5}{4^2}x^3+\dots$ is convergent provided $|x|<1$.

Examine the cases when $x=1$ and $x=-1$.

11. Show that the series whose nth term is $\frac{1}{n\,.\,2^n}$ is convergent and find its sum by comparing it with the series for $\log(1-x)$.

12. Discuss the convergence of the following series:

(i) $\frac{5}{1\cdot3}x-\frac{1}{2\cdot4}\frac{5^2}{3^2}x^2+\frac{1}{3\cdot5}\frac{5^3}{3^3}x^3-\frac{1}{4\cdot6}\cdot\frac{5^4}{3^4}x^4+\dots$;

(ii) $1+\frac{5}{3}x+\frac{1\cdot4}{2!}\frac{5^2}{3^2}x^2+\frac{1\cdot4\cdot7}{3!}\frac{5^3}{3^3}x^3+\dots$.

Also identify the series.

Test Papers C

C.I

1. (i) Find the equation whose roots are the squares of the roots of the equation
$$2x^2+3x-1=0.$$
(ii) Show that if x is real the function
$$\frac{4x-3}{x^2+1}$$
can only take values in a certain range and find this range. (L.)

2. If x is small so that x^3 and higher powers of x may be neglected, express the function

$$\frac{\sqrt{(4+x)}}{1-2x+(1+3x)^{\frac{2}{3}}}$$

in the form $a+bx+cx^2$. (O. & C.)

3. Solve the equation

$$2x^3-7x^2+10x-6=0$$

given that $x=1-i$ is one of its roots. (O. & C.)

4. If $e^x=e^{3x}-\frac{1}{4}$ show that a first approximation gives $x=\frac{1}{8}$ and a second (neglecting x^3) gives $x=\frac{3}{32}$. Find the third approximation neglecting all powers of x above x^3.

5. Assuming the expansion for $\log_e(1+x)$ in ascending powers of x, prove that

$$\log_e\sqrt{\left(\frac{1+x}{1-x}\right)}=x+\frac{x^3}{3}+\frac{x^5}{5}+\dots,$$

and, when $0<\theta<\frac{1}{2}\pi$, deduce that

$$\sin\theta+\tfrac{1}{3}\sin^3\theta+\tfrac{1}{5}\sin^5\theta+\dots=\log_e(\tan\theta+\sec\theta).$$

(O. & C.)

6. Show how to represent the sum and difference of two complex numbers z_1, z_2 in an Argand diagram.

If $|z_1|=|z_2|$, prove from geometrical considerations or otherwise that $\dfrac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

Show that the equation $z^3+8=0$ has one real root and two complex roots. Give the modulus and argument of each root and show the roots in an Argand diagram.

Find the value of $\dfrac{(1-i)^2(\sqrt{3}+i)}{1-\sqrt{3}i}$. (O. & C.)

C.II

1. In how many ways can a cricket eleven be selected from fifteen players? If ten of the players are right-handed and five left-handed, find how many of the possible elevens will include (i) two and only two left-handers, (ii) two or more left-handers.

2. (i) Prove that, if the equations

$$x^2+x+p=0,\quad qx^2-x+1=0$$

have a common root, then

$$(pq-1)^2+(p+1)(q+1)=0.$$

(ii) Prove that the sum of the first n terms of the geometrical progression a, ar, ar^2, ... is $a(r^n-1)/(r-1)$.

The sum of the first five terms of the geometrical progression is 5; the sum of the fifth to the ninth terms (both inclusive) is 80. Prove that $r = \pm 2$, and find the two possible values of a. (O. & C.)

3. Express the function

$$\frac{2}{x(x+1)(x+2)}$$

in partial fractions and hence prove that the sum of the first n terms of the series

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \ldots$$

is $$\frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$ (O. & C.)

4. (i) Write down the rth term in the expansion of $(1+x)^n$. If the coefficients of x^r, x^{r+1} and x^{r+2} in the expansion of $(1+x)^{14}$ are in arithmetical progression, find the possible values of r.

(ii) If the square and higher powers of x can be neglected prove that $\dfrac{(4+x)^{\frac{1}{2}}(1+3x)^{\frac{1}{3}}}{1+2x} = 2 - \dfrac{7x}{4}$. (L.)

5. Show that the limits as $n \to \infty$ of $\left(1+\dfrac{x}{n}\right)^n$ and of $\left(1+\dfrac{1}{n}\right)^{nx}$ are the same.

6. Use de Moivre's theorem to prove that, if

$$2 \cos \theta = x + \frac{1}{x}$$

then $$2 \cos n\theta = x^n + \frac{1}{x^n},$$

Hence, or otherwise, solve the equation

$$5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0.$$ (O. & C.)

C.III

1. Prove that $a^2b^2 + b^2c^2 + c^2a^2$ is a perfect square if either $a+b+c=0$ or if any one of a, b, and c is the sum of the other two.

Hence, or otherwise, find the square root of

$$(x-1)^2(4x+1)^2 + (4x+1)^2(3x+2)^2 + (3x+2)^2(x-1)^2.$$ (L.)

2. Using the relation $i = \frac{1}{2}(1+i)^2$ find (i) four complex factors, (ii) two real quadratic factors for $z^4 + 1$.

3. The sum to infinity of a series of numbers in geometric progression is 57, and the sum to infinity of a series with the same first term but with a common ratio the cube of the original is 27. Find the first series.

Sum to infinity the series

$$1+(1+a)r+(1+a+a^2)r^2+(1+a+a^2+a^3)r^3+\dots,$$

stating the range of values of a and of r for which this is possible.

4. If a, b, c are real and $p+iq$ is a root of $az^2+bz+c=0$, prove that $2apq+bq=0$ and $aq^2=ap^2+bp+c$.

5. (*a*) Find the least number of years in which money invested at 4% per annum compound interest will double itself.

(*b*) If $2\log_{10}x+5\log_{10}y=1{\cdot}232$ express x in terms of y in a form not involving logarithms.

(*c*) Solve the equation $3^{4x+3}-13\times 3^{2x+1}+4=0$. (L.)

6. Sum the series :

(i) $nc_0-(n-1)\,c_1+\dots+(-)^{n-1}\,c_{n-1}$

where $(1+x)^n\equiv c_0+c_1x+\dots c_nx^n$;

(ii) $1+\frac{1}{6}+\frac{1\,.\,4}{1\,.\,2}\cdot\frac{1}{6^2}+\frac{1\,.\,4\,.\,7}{1\,.\,2\,.\,3}\cdot\frac{1}{6^3}+\dots$;

(iii) $\sum\limits_{2}^{\infty}\frac{n^2-1}{n!}$. (L.)

C.IV

1. Show that, if $y=\frac{x^2-5}{(x+1)(x-2)}$, y cannot lie between $\frac{10}{9}$ and 2 when x is real.

Draw a rough graph of the function and indicate clearly how the function behaves for large (positive and negative) values of x. (B.)

2. (i) Prove that, if two polynomials $P(x)$ and $Q(x)$ have a common linear factor $x-p$, then $x-p$ is a factor of the polynomial

$$[P(x)-Q(x)].$$

Hence prove that, if the equations

$$ax^3+4x^2-5x-10=0,\quad ax^3-9x-2=0$$

have a common root, then $a=2$ or 11.

(ii) Prove that, if $x+\frac{1}{x}=y+1$, then

$$\frac{(x^2-x+1)^2}{x(x-1)^2}=\frac{y^2}{y-1}.$$

Hence solve the equation

$$(x^2-x+1)^2-4x(x-1)^2=0.$$ (O. & C.)

3. Find the first three terms and the coefficient of x^r in the expansion of

$$\log_e(1-x-2x^2)$$

in ascending powers of x, and state the range of values for which the expansion is valid. (O. & C.)

4. Expand $(1-\frac{1}{4}x^4)^{-4}$ in ascending powers of x as far as the term containing x^{12}.

The function

$$e^{-x}-\frac{1-x}{(1-x^2)^{\frac{1}{2}}(1-x^3)^{\frac{1}{3}}},$$

is expanded in ascending powers of x in the form

$$ax+bx^2+cx^3+dx^4+\dots .$$

Prove that the four coefficients a, b, c, d are all zero.

5. Define $\sinh x$ and $\cosh x$ in terms of exponential functions.

Prove that $\cosh^2 x-\sinh^2 x=1$.

Find the real values of x satisfying the equation

$$4\sinh^2 x=6-3\cosh x,$$

giving your answer in logarithmic form. (O. & C.)

6. Show on the Argand Diagram the circle inside which z must lie so that the series

$$\frac{2z}{z+3}+\left(\frac{2z}{z+3}\right)^2+\left(\frac{2z}{z+3}\right)^3+\dots$$

may be convergent.

C.V

1. (i) Given that $(x-\alpha)(x-\beta)(x-\gamma)\equiv x^3+px^2+qx+r$, express p, q, r in terms of α, β, γ.

Solve the equation

$$x^3-21x^2+126x-216=0,$$

given that the roots are in geometric progression.

(ii) Solve the simultaneous equations

$$3x^2+15xy-56y^2+56=0,$$
$$2x^2+9xy-33y^2+28=0.$$

(O. & C.)

2. In the same diagram sketch the graphs of the functions

$$\frac{(x+1)(x-3)}{x-4} \text{ and } \frac{-2}{x},$$

for values of x from -3 to 4.

Prove that the abscissa of the point of intersection of the two graphs is a root of the equation

$$x^3-2x^2-x-8=0.$$

Prove (do not merely derive the result from your graph) that the maximum ordinate of the first graph within the range -3 and $+3$ for x is $6-2\sqrt{5}$. (O. & C.)

3. Write down the first four terms in the expansion of $\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}$ by the Binomial Theorem.

Show that

$$\left(1+\frac{x}{3}\right)^{-\frac{1}{2}} . e^{-x} = 1 - \frac{7x}{6} + \frac{17x^2}{24} + \dots,$$

and find the coefficient of x^3 in this expansion. (B.

4. Show that $\frac{1}{2} \log \frac{n^2}{n^2-1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$ where $x = \frac{1}{2n^2-1}$ and $n>1$.

5. (i) Prove that one of the values of $\tan^{-1}(\frac{1}{7}) + 2\tan^{-1}(\frac{1}{3})$ is $45°$.
(ii) Write down the series for $\tan^{-1} x$ and use the result (i) to calculate the value of π to four figures.

6. Define the modulus and the amplitude (argument) of a complex number with reference to an Argand Diagram.

Show that the equation $z^3 = 2$ has one real root and two complex roots. Give the modulus and amplitude of each root.

Find the modulus and amplitude of the complex number

$$(\sqrt{3} - i)/(1+i).$$ (O. & C.)

C.VI

1. Prove that it is possible to find n consecutive odd numbers whose sum is n^3, and find the middle number or the two middle numbers, according as n is odd or even.

2. In the Argand Diagram show that the point given by $4z_2 - 3z_1$ is collinear with the points given by z_1 and z_2, and explain where it lies on this line. [$z_1 \equiv x_1 + iy_1$, and so for z_2.]

3. Use the binomial series to write down the first four terms of the expansion of $(1+y)^{-\frac{1}{2}}$ in a series of ascending powers of y.

Hence find, in terms of $\cos\theta$, the coefficients c_1, c_2, c_3 in the expansion of

$$(1 - 2x\cos\theta + x^2)^{-\frac{1}{2}} \text{ in the form } 1 + c_1x + c_2x^2 + c_3x^3 + \dots .$$

Prove that, when $\theta = 0$, every coefficient in the series is equal to $+1$.

[You may assume throughout that the expansions are valid.]

4. Write down the expansion of $\log_e \frac{1+x}{1-x}$ in a series of ascending powers of x, and prove that it is convergent for $-1<x<+1$.

Expand $\log_e\left(1+\frac{1}{n}\right)$ in ascending powers of $\frac{1}{2n+1}$, stating the necessary restrictions on the value of n and show that, if $n>1$,

$$\frac{2}{2n+1} < \log_e\left(1+\frac{1}{n}\right) < \frac{1}{n}.$$ (B.)

5. Prove that

$$2!+\frac{3!}{(1!)^2}+\frac{4!}{(2!)^2}+\frac{5!}{(3!)^2}+\ldots \text{ to infinity} = 7e. \qquad \text{(B.)}$$

6. Express $\dfrac{7x-x^2}{(1-x)(1+2x)(1+x^2)}$ in partial fractions.

For what values of x can this be expanded in ascending powers of x? If it can be expanded thus, show that the coefficient of x^4 is numerically double that of x^3. (B.)

C.VII

1. (i) If 1, ω, ω^2 are the cube roots of unity, prove that if n is not a multiple of 3, then $1+\omega^n+\omega^{2n}=0$.

What is the value of $1+\omega^n+\omega^{2n}$ if n is a multiple of 3?

(ii) Factorise z^2+z+1 and u^2+uv+v^2 in complex algebra.

2. (i) Being given that $x^2+y^2=1$ and that $x/y=\lambda$, express $7x^2+2xy-3y^2$ in terms of λ. (L.)

(ii) Being given that a, b, c are positive numbers, prove that

$$3(a^3+b^3+c^3)-(a^2+b^2+c^2)(a+b+c)=\Sigma(a^2-b^2)(a-b)$$

and deduce that it is positive.

3. Show that, if x_1 is an approximate value of $\sqrt{N}$, then x_2, x_3, ... are successively improved approximations if

$$x_2=\tfrac{1}{2}(x_1+N/x_1), \quad x_3=\tfrac{1}{2}(x_2+N/x_2), \ldots .$$

Find, by this method of approximation, the square root of 38 correct to three decimal places. (B.)

4. (i) Divide x^3-2-2i by $x+1-i$, and hence prove that the three cube roots of $2+2i$ are $i-1$ and $\tfrac{1}{2}\{1-i\pm\sqrt{3}(1+i)\}$.

(ii) What is the locus of z on the Argand Diagram if $\left|\dfrac{z-z_1}{z-z_2}\right|=2$? (C.)

5. (i) State precisely the ranges of values of x for which the expansions of $\log_e(1+x)$ and $\log_e(1-x)$ are valid.

Prove that, for a certain range of values of x which is to be stated, the sums of the two series

$$x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots$$

$$\frac{x}{1+x}+\frac{1}{2}\left\{\frac{x}{1+x}\right\}^2+\frac{1}{3}\left\{\frac{x}{1+x}\right\}^3+\frac{1}{4}\left\{\frac{x}{1+x}\right\}^4+\ldots$$

are equal.

(ii) Show that the difference between the value of e and the sum of the first n terms in its expansion lies between

$$\left\{1+\frac{1}{n+1}\right\}\frac{1}{n!} \quad \text{and} \quad \left\{1+\frac{1}{n}\right\}\frac{1}{n!},$$

and hence show that the first eight terms of the expansion give the value of e correct to four places of decimals. (L.)

6. (i) State the series for $\log(1+x)$, where $|x|<1$, and deduce the series for $\log\dfrac{1+x}{1-x}$.

How can this series be used to deduce log 37 from log 6?

(ii) Assuming that $\log x=\int_1^x \frac{dt}{t}$, show from a sketch graph that

$$\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}>\log\tfrac{7}{4}>\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}.$$

[The logarithms in this question are to base e.] (L.)

C.VIII

1. Determine the cube roots of unity by a construction of an Argand Diagram, and show that the two complex roots are such that each is the square of the other. Calling these roots ω and ω^2, find linear factors for a^3+b^3 and for $a^3+b^3+c^3-3abc$.

2. If S_k is the sum of k terms of the geometric progression whose first term is a and common ratio r, obtain an expression for the sum T_n of n terms of the series

$$T_n=S_1+S_2+\ldots+S_n,$$

and verify that

$$rS_n+(1-r)T_n=na.$$

If $r=\frac{1}{2}$ and $S_6=\frac{7}{16}$, find the value of T_6. (L.)

3. Write down the $(r+1)$th term of the binomial expansion of

$$\left(x^2-\frac{3}{x}\right)^n.$$

Show that there is a term which is positive and independent of x if n is a multiple of 3.

Find the numerically greatest term in the expansion for $x=\frac{3}{2}$ and $n=9$. (L.)

4. Prove that the coefficients of x^n in the binomial expansions of $(1-x)^{-2}$ and $(1-x)^{-3}$ are $n+1$ and $\frac{1}{2}(n+1)(n+2)$.

By putting $x=\frac{1}{2}$ in each of these two expansions prove that

$$\frac{1^2}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\ldots+\frac{n^2}{2^n}+\ldots=6. \quad \text{(L.)}$$

5. (i) Express
$$\frac{x+1}{(x-1)(x^2+1)}$$
in partial fractions.

(ii) Expand
$$\log_e \frac{(1-3x)^2}{(1-2x)^3}$$
as far as the term in x^4 and find the coefficient of x^r.

State the range of values of x for which the expansion is valid. (L.)

6. Find the sum to infinity of:

(i) $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \ldots$;

(ii) $1 + \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} + \frac{1}{7\cdot 3^3} + \ldots$.

CHAPTER XIII

INEQUALITIES

Inequalities

The statement "a is greater than b" means that $a-b$ is positive. Thus -3 is greater than -5. This is written $-3>-5$.

The symbolic statement "$-5<-3$" means "-5 is less than -3" and $-5-(-3)$ is negative.

The signs $>$ and $<$ are also used for "greater than" and "less than" without the verb; $\ngtr$ means "is not greater than", and so it is equivalent to "is $\leqslant$ to".

To denote that a is numerically greater than b, the modulus notation is used, and we write $|a|>|b|$. Thus $|-5|>|-3|$.

The process of adding the same number to each side can be applied to an inequality just as to an equation.

Thus if $x-5<a$, it follows that $x<a+5$.

In general if $a>b$, then $a+c>b+c$ for all values of c, positive or negative.

Again, if $\frac{x}{2}>a$, it follows that $x>2a$.

Both sides of an inequality may be multiplied by any *positive* number, but multiplication by a *negative* number changes $>$ into $<$;

for example, if $\dfrac{x}{a-b}>c$, then if $a>b$, $x>(a-b)c$;

but if $a<b$, $x<(a-b)c$.

Suppose $$3x+7>5x+2,$$
then $$(3-5)x>-5,$$
and this is changed into $2x<5$.

Example I. For what values of x is $\frac{5}{2}x+\frac{3}{4}<2x+\frac{1}{4}$?

From the inequality (multiplying by 4),

$$10x+3<8x+1.$$
$$\therefore\ 2x<-2, \quad x<-1.$$

The inequality is satisfied for all values of x less than -1.

Example II. For what values of x is $2<\dfrac{x-7}{x-2}<3$?

If $x>2$, i.e. if $x-2$ is positive, we require

$$2(x-2)<x-7<3(x-2).$$

For $2x-4<x-7$ we need $x<-3$, which contradicts $x>2$.

$$\therefore\ x>2 \text{ is not possible.}$$

Again, if $x<2$, i.e. if $x-2$ is negative, we require

$$2(x-2)>x-7>3x-6.$$

For $2x-4>x-7$ we get $x>-3$ and for $x-7>3x-6$ we get $x<-\frac{1}{2}$. Hence we see that the only values possible are in the range $-3<x<-\frac{1}{2}$.

Inequalities and Graphs

To discuss the inequality $ax+b>cx+d$ we can change it into

$$(a-c)x+(b-d)>0$$

and consider what part of the graph of $y=(a-c)x+(b-d)$ is above the x-axis.

In Example II above, the question may be asked thus:

> "For what part of the graph of $y=\dfrac{x-7}{x-2}$ is the value of y between 2 and 3?"

Quadratic Inequalities

Many quadratic inequalities have been discussed previously from the graphic point of view.

To ask "for what values of x is $ax^2+bx+c>a'x^2+b'x+c'$?" is to ask "for what values of x is the graph of

$$y=(a-a')x^2+(b-b')x+c-c'$$

above the x-axis?"

Thus $\qquad 3x^2+5x+10<x^2+18x-5$

if $\qquad 2x^2-13x+15<0$

or $\qquad (2x-3)(x-5)<0.$

The graph of $y=(2x-3)(x-5)$ will be below the x-axis (or the factors of y will have opposite signs) if

$$x \text{ lies between } 1\tfrac{1}{2} \text{ and } 5.$$

Again, limits on possible values of $\dfrac{\text{quadratic}}{\text{quadratic}}$ have been discussed in Chapter III.

For example, $y\equiv\dfrac{x(x-1)}{(x-2)(x-3)}$ can take no value such that $(y+7)^2<48$. (No. 9 below.)

Examples 118

1. For what range of values of x are the functions $\frac{2}{3}x-2$ and $3-\frac{3}{5}x$ (i) both positive; (ii) the first positive and the second negative; (iii) the first greater than the second; (iv) the first greater than twice the second?
2. For what values of x does x^2-5x+4 lie between -2 and $+2$?
3. For what values of x is the function $14x-20-2x^2$ (i) greater than 5; (ii) between 1 and 4?
4. Find the ranges of x for which $5-\dfrac{2}{x-4}>0$.
5. When is $3+\dfrac{5}{x+2}<0$?
6. For what values of x do we have $-1<\dfrac{2x+3}{x-1}<1$?
7. Show that $4x/(x^2+2x+2)$ is always less than 1.
8. Show that $\dfrac{x^2-4pq}{x-(p+q)}$ cannot lie between p and q.
9. Prove that if
$$y=\frac{x(x-1)}{(x-2)(x-3)},$$
then y can take no value such that $(y+7)^2<48$.
10. Prove that $\dfrac{4x^2+5x+1}{x^2+2x+2}$ lies between $4\frac{1}{2}$ and $-\frac{1}{2}$.
11. Prove that $\dfrac{x^2+5x-8}{x^2+4x-5}$ cannot lie between $1\frac{1}{2}$ and $1\frac{1}{18}$.
12. Prove that $\dfrac{2x^2-14x+11}{2x^2-2x+5}$ lies between 3 and -1.

Inequalities not involving a Variable

Many inequalities are deduced from the simple fact that

if two factors have the same sign their product is positive,

and in particular *all squares are positive.*

In what follows, all letters stand for positive numbers.

Example I. A.M.$>$G.M. for two numbers a, b, which are unequal. Since squares are positive $(a-b)^2>0$;

$$\therefore\ a^2+b^2>2ab;\quad \therefore\ (a+b)^2>4ab;$$
$$\therefore\ \left(\frac{a+b}{2}\right)^2>ab.$$

$\therefore\ \dfrac{a+b}{2}>\surd(ab)$; that is the A.M. of a and $b>$ their G.M.

This result must be remembered.

Example II. Unless a, b, c are all equal, prove that

$$a^2+b^2+c^2>bc+ca+ab.$$

Because $2(a^2+b^2+c^2-bc-ca-ab)=(b-c)^2+(c-a)^2+(a-b)^2$, and as the R.H.S. is positive, the required result follows.

Alternatively, we may use Example I and say $\dfrac{a^2+b^2}{2}>ab$ and add to this two similar statements to give the result.

Example III. Prove that $2(a^3+b^3+c^3)\geqslant a^2(b+c)+b^2(c+a)+c^2(a+b)$.

The sides are equal if $a=b=c$.

If not, L.H.S. $-$ R.H.S.

$$=(b^3+c^3-b^2c-bc^2)+(c^3+a^3-c^2a-ca^2)+(a^3+b^3-a^2b-ab^2)$$
$$=(b^2-c^2)(b-c)+(c^2-a^2)(c-a)+(a^2-b^2)(a-b).$$

In each of these products the two factors must have the same sign, and so their product is positive.

This proves the required result.

Example IV. If a, b are unequal, prove that

$$\frac{a^{m+n}+b^{m+n}}{2}>\frac{a^m+b^m}{2}\cdot\frac{a^n+b^n}{2}.$$

We have to prove $2(a^{m+n}+b^{m+n})-(a^{m+n}+a^nb^m+a^mb^n+b^{m+n})>0$, which is the same as $(a^m-b^m)(a^n-b^n)>0$.

This is true, since the two factors must have the same sign.

N.B In all these examples of inequalities, the letters must not be all equal. If they are all equal the $>$ sign is replaced by the $=$ sign.

Examples 119

1. Use A.M.>G.M. to prove that

$$(a+b)(b+c)(c+a)>8abc.$$

2. Prove that $a^7+b^7\geqslant a^2b^2(a^3+b^3)$.
3. Prove that $(a_1{}^2+a_2{}^2)(b_1{}^2+b_2{}^2)\geqslant(a_1b_1+a_2b_2)^2$.
4. Prove that $(a_1{}^2+a_2{}^2+a_3{}^2)(b_1{}^2+b_2{}^2+b_3{}^2)\geqslant(a_1b_1+a_2b_2+a_3b_3)^2$.
5. Prove that $\frac{1}{2}(a^3+b^3)>\left(\dfrac{a+b}{2}\right)^3$ unless $a=b$.
6. Prove that $a^3+b^3+c^3\geqslant 3abc$.

   [Hint. $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$. What is the other factor?]
7. Use Example IV to prove that

$$\frac{a^5+b^5}{2}>\frac{a^3+b^3}{2}\cdot\frac{a^2+b^2}{2}>\left(\frac{a^2+b^2}{2}\right)^2\cdot\frac{a+b}{2}>\left(\frac{a+b}{2}\right)^5.$$

8. Prove that $4(a_1{}^{10}+a_2{}^{10})>(a_1{}^5+a_2{}^5)(a_1{}^3+a_2{}^3)(a_1{}^2+a_2{}^2)$.

9. Prove that $3a^2+14b^2>12ab$.

[Hint. Use sums of squares.]

10. Show that (i) $a^3+8b^3+27c^3 \geqslant 18abc$.

[Use the result of No. 6.]

(ii) $9a^3+28b^3+126c^3>93abc$.

When does the equality sign apply in (i), and why is it impossible for an equality sign to appear in (ii)?

Sum and Product

The identity

$$(a+b)^2=4ab+(a-b)^2$$

together with the fact that the least value of $(a-b)^2$ is 0,

which happens if $a=b$,

proves that

If the sum of two numbers is given, their product is greatest when the numbers are equal, ...(i)

and that

If the product of two numbers is given, their sum is least when the numbers are equal. ....(ii)

A little thought will show that either of (i) (ii) follows from the other.

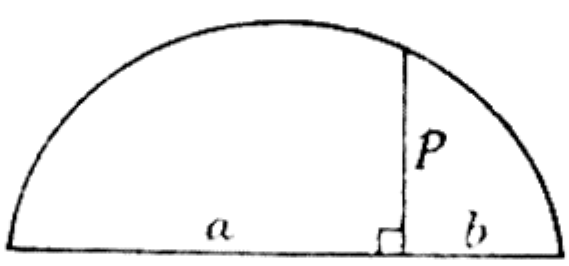

FIG. 52

The result (i) also follows from Fig. 52, which shows a semi-circle drawn on a line of length $a+b$ with a perpendicular p drawn as shown; for either the intersecting-chords theorem or the direct use of similar triangles proves that $p^2=ab$, and p is seen to be greatest when drawn from the centre of the circle.

Examples 120

1. If $2x+3y=4$, prove that the greatest value of xy is $\frac{2}{3}$.

2. If $ab=27$, what is the least value of $4a+5b$?

3. Given $5x+y=7$, what is the greatest value of xy?

4. If $3x+4y=5$, what is the least value of $9x^2+16y^2$?

5. Show that the minimum value of $x+\dfrac{3}{x}$ is $2\sqrt{3}$.

[Such questions may, of course, be answered using Calculus without *much* extra trouble.]

Generalisation

The sum and product theorem above applies to more than two numbers, e.g. given $a+b+c$, then the product abc is greatest if $a=b=c$.

For if $a+b+c=3s$, start with each number equal to s and then take $r(<s)$ from one of the numbers and split it into 2 portions p and q, which are added to the other two, so that the sum is unaltered.

Now the product is $(s+p)(s+q)(s-r)$ where $p+q=r$.

This product expanded is $s^3+(p+q-r)s^2-(pr+qr-pq)s-pqr$,

$$\text{i.e. } s^3-[(p+q)^2-pq]s-pqr, \text{ which is } <s^3.$$

$\therefore$ the product is greatest when the numbers are equal if the sum is given. We shall assume this to be true generally, and give a complete proof later (see p. 322).

Example I. If $p+q+r=6$, find the greatest value of $p^3q^2r^2$.

Solution. $p^3q^2r^2$ is maximum if $\left(\frac{p}{3}\right)^3\left(\frac{q}{2}\right)^2\left(\frac{r}{2}\right)^2$ is maximum.

This last product consists of 7 terms whose sum is

$$3\left(\frac{p}{3}\right)+2\left(\frac{q}{2}\right)+2\left(\frac{r}{2}\right)=p+q+r=6.$$

$\therefore$ the product is maximum if the terms are equal,

$$\text{i.e. if } \frac{p}{3}=\frac{q}{2}=\frac{r}{2}=\frac{p+q+r}{7}=\frac{6}{7}.$$

$\therefore$ maximum value of $p^3q^2r^2$ is $\dfrac{18^3 \cdot 12^2 \cdot 12^2}{7^7}$ $(\simeq 146{\cdot}9)$.

Examples 121

1. If $2a+3b+c=18$, show that the maximum value of abc is 36.
2. If $a+b+c=12$, find the maximum value of a^2bc.
3. If $p+q=18$ show that the maximum value of p^5q^4 is when $p=10$, $q=8$ and $\simeq 4{\cdot}096\times 10^8$.
4. If $xy=36$ and $x\geqslant 4$, what are the least and greatest values of $(x+y)$?
5. Given a circle centre O, A a point outside it, and AT a tangent; if APQ is any line meeting the circle in P and Q and ON is the perpendicular to PQ from O, show $AP+AQ=2AN>2AT$.

   Deduce from $AP \cdot AQ=AT^2$ that if the product of two numbers is given their sum will be least when they are equal.
6. If $p+q+r=3$ show that the maximum value of $p^4q^3r^2$ is $2^{10} \cdot 3^{-6}$.
7. If $p^2qr=324$ show that the minimum value of $p+q+r$ is 12.
8. If $2x+3y=4$ find the greatest value of x^3y^2.

9. If $a+b+c=12$ show that the greatest value of $a^4b^3c^2$ is $4^{14} \cdot 3^{-6}$.

10. Use the fact that $\left(\frac{p+q}{2}\right)^4 \geqslant p^2q^2$ to show that

$$\left(\frac{a_1+a_2+a_3+a_4}{4}\right)^4 \geqslant a_1a_2a_3a_4$$

by putting $p=\frac{a_1+a_2}{2}$ and $q=\frac{a_3+a_4}{2}$.

Extend to establish that

$$\left(\frac{a_1+a_2+\ldots+a_8}{8}\right)^8 \geqslant a_1a_2 \ldots a_8.$$

Generalisations

The results of Examples I and IV (p. 318) can be generalised by increasing the number of letters involved from 2 to 3, 4, ..., n.

A.M.>G.M.

The result for n letters (the letters not being all equal) is

$$\frac{1}{n}(a_1+a_2+a_3+\ldots+a_n) > \sqrt[n]{(a_1a_2a_3 \ldots a_n)},$$

the left side being *defined* as the *Arithmetic Mean* and the right side as the *Geometric Mean* of the n letters.

The case of 3 letters will first be considered, the method used being one that admits of complete generalisation.

To prove that $\frac{1}{3}(a+b+c) > \sqrt[3]{abc}$ unless $a=b=c$.

Let $abc=G^3$ so that G is the G.M.

Now if a and c are greatest and least of a, b, c, so that $a>G>c$, replace them by G and $\frac{ac}{G}$ so that the numbers are now $G, b, \frac{ac}{G}$.

[This replacement would also be made when

$$a=b>c \text{ or when } a>b=c.]$$

By this change the *increase* in the sum of the numbers is

$G+\frac{ac}{G}-a-c$, i.e. $(G-a)(G-c)/G$, which is <0.

$\therefore$ there is a *decrease* in the A.M. with no change in the G.M.

Since $b \times \frac{ac}{G} = \frac{abc}{G} = G^2$, so that if $b \neq \frac{ac}{G}$ then $b>G>\frac{ac}{G}$ or $\frac{ac}{G}>G>b$.

In either case replace b and $\frac{ac}{G}$ by G and $\frac{abc}{G^2}$.

This second *increase* in the sum of the numbers is

$$G+\frac{abc}{G^2}-b-\frac{ac}{G}, \text{ i.e. } (G-b)\left(G-\frac{ac}{G}\right)\Big/G, \text{ which is } <0.$$

Again there is a *decrease* in the A.M. to the value G, since the numbers are now G, G, G.

$\therefore$ the original A.M.$>$the final A.M., i.e. $\frac{1}{3}(a+b+c)>\sqrt[3]{abc}$.

Note. There is an alternative way of approaching this theorem by increasing the G.M. instead of reducing the A.M. In this way we replace a and c each by $\frac{1}{2}(a+c)$. This does not alter the A.M., but it increases the G.M. since $ac<\{\frac{1}{2}(a+c)\}^2$.

If we could continue this process till the numbers were all equal, it would prove the theorem, but in the case of three numbers, the numbers—though they get nearly equal—never become equal.

For example, start with 1, 4, 9; the first few stages are

$$5,\ 5,\ 4\ ;\quad 5,\ 4\tfrac{1}{2},\ 4\tfrac{1}{2}\ ;\quad 4\tfrac{3}{4},\ 4\tfrac{3}{4},\ 4\tfrac{1}{2}\ ;\quad 4\tfrac{5}{8},\ 4\tfrac{5}{8},\ 4\tfrac{3}{4}\ ;$$

and there is always the "odd man out".

This way works if n is a power of 2. It will be found to work for four numbers.

A.M.>G.M. General Case

Given the n numbers $a_1, a_2, a_3, \ldots, a_n$.

Let G be their G.M. and p, q the largest and smallest of the numbers.

Replace p and q by G and $\dfrac{pq}{G}$. This leaves the product unaltered but diminishes the sum.

Repeat this process, always using the greatest and least numbers.

At each stage one more of the numbers has been replaced by G, and finally the stage is reached when the numbers have been replaced by $n-1$ G's and $\dfrac{a_1a_2\ldots a_n}{G^{n-1}}$ which $=G$.

Since at each stage the G.M. is unaltered while the A.M. is diminished, and since at the last stage they become equal it follows that originally

$$\text{the A.M.}\quad \frac{a_1+a_2+\ldots+a_n}{n}>(a_1a_2\ldots a_n)^{\frac{1}{n}}.$$

(If the numbers are all equal the sign $>$ is replaced by $=$.)

Example IV (p. 319) generalised

The general result is that for the k letters $a_1, a_2, \ldots, a_k$

$$\frac{a_1^{m+n}+a_2^{m+n}+\ldots+a_k^{m+n}}{k} > \frac{a_1^m+a_2^m+\ldots+a_k^m}{k} \cdot \frac{a_1^n+a_2^n+\ldots+a_k^n}{k}$$

unless the a's are all equal.

We begin by considering the 3 letters a, b, c, and prove that

$$\frac{a^{m+n}+b^{m+n}+c^{m+n}}{3} > \frac{a^m+b^m+c^m}{3} \cdot \frac{a^n+b^n+c^n}{3},$$

the letters representing positive numbers and a, b, c not all equal.

Solution. We have to prove that

$$3(a^{m+n}+b^{m+n}+c^{m+n}) > (a^m+b^m+c^m)(a^n+b^n+c^n).$$

$$\begin{aligned} \text{L.H.S.} - \text{R.H.S.} &= 2(a^{m+n}+b^{m+n}+c^{m+n}) - \Sigma(a^m b^n + a^n b^m) \\ &= \Sigma(a^m - b^m)(a^n - b^n). \end{aligned}$$

This is positive since the two factors in each group have the same sign.

Note. The case for 4 letters should be worked next.

In the general case for k letters, we have to prove that

$$(k-1)\Sigma a_1^{m+n} - \Sigma(a_1^m a_2^n + a_2^m a_1^n) > 0$$

where the first Σ applies to the k letters and the second Σ, being for each pair of letters, implies $\frac{1}{2}k(k-1)$ pairs, so we get

$$\Sigma(a_1^{m+n} + a_2^{m+n} - a_1^m a_2^n - a_2^m a_1^n) > 0,$$

the Σ being for all pairs.

This is $\Sigma(a_1^m - a_2^m)(a_1^n - a_2^n) > 0$, which is true.

(To master this, the student should write it out in full.)

Examples 122

1. Write out the proof that the A.M. > the G.M. for four positive numbers, not all equal.
2. Write out the proof for four positive numbers a, b, c, d, not all equal, that

$$\frac{a^{m+n}+b^{m+n}+c^{m+n}+d^{m+n}}{4} > \frac{a^m+b^m+c^m+d^m}{4} \cdot \frac{a^n+b^n+c^n+d^n}{4}.$$

3. If a, b, c are positive and not all equal, prove that

(i) $(a^2+b^2+c^2)(a+b+c) > 9abc$;

(ii) $9(a^3+b^3+c^3) > (a+b+c)^3$.

4. If a is positive, prove that

$$1+a+a^2+a^3+a^4>5a^2.$$

5. Prove that $a^{x-2y}+a^{x-y}+a^{x+y}+a^{x+2y}>4a^x$.
6. If $0<x<1$ and $0<y<1$, show that $0<x+y-xy<1$.
7. If $a^2+b^2=1$ and $c^2+d^2=4$, prove $abcd<1$.
8. Prove that $(a+b+c+d)^5<64(a^3+b^3+c^3+d^3)(a^2+b^2+c^2+d^2)$ if a, b, c, d are unequal positive numbers.
9. Show that a triangle with given perimeter $2s$ will have maximum area when it is equilateral, when the area is $s^2/3\sqrt{3}$.
10. With the usual notation associated with $\triangle ABC$, show that

$$\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)\geqslant 3(s/\triangle^2)^{\frac{1}{3}}.$$

11. Show that $(2n-1)!/2[(n-1)!]<n^n$.
12. Show that $\sum_{r=1}^{n} r^n>n\left(\frac{n+1}{2}\right)^n$.
13. Use the result $\log x\leqslant x-1$ with equality only if $x=1$. [See Chap. XII, pp. 293.] Apply this to the n quantities $\frac{a_1}{A}, \frac{a_2}{A}, \ldots \frac{a_n}{A}$, where A is the arithmetic mean of $a_1, a_2, \ldots, a_n$, to prove the result:

$$\log\frac{a_1a_2\ldots a_n}{A^n}\leqslant\frac{a_1+a_2+\ldots+a_n}{A}-n,\quad \text{i.e.} \leqslant 0.$$

Deduce that the G.M. of $a_1, a_2 \ldots a_n$ is less than A unless all the a's are equal.

Inequalities using integration

Frequently we can make use of integration to establish an inequality; the method used being dependent on integration giving the area under a curve.

In Fig. 53, if the continuous curve is $z=t^2$ then the area under the curve between P and Q is $\int_1^x t^2\,dt$, i.e. $\frac{x^3}{3}-\frac{1^3}{3}$.

Now area $LQNM>$ area under the curve $>$ area $PHNM$;

$$\therefore\ x^2(x-1)>\frac{x^3}{3}-\frac{1}{3}>1^2(x-1),$$

$$\text{i.e. } 3x^2(x-1)>x^3-1>3(x-1).$$

These enable us to say that if $x>1$, then $x^3>\frac{1}{2}(3x^2-1)$ and

$$x^3>3x-2.$$

Similarly if the dotted curve is that of $z=\frac{1}{t}$,

$$\text{Area } PHNM > \int_1^x \frac{dt}{t} > \text{area } L'Q'NM,$$

$$\text{i.e. } 1 \,.\, (x-1) > \log_e x > \frac{1}{x}(x-1) \quad \text{if } x>1.$$

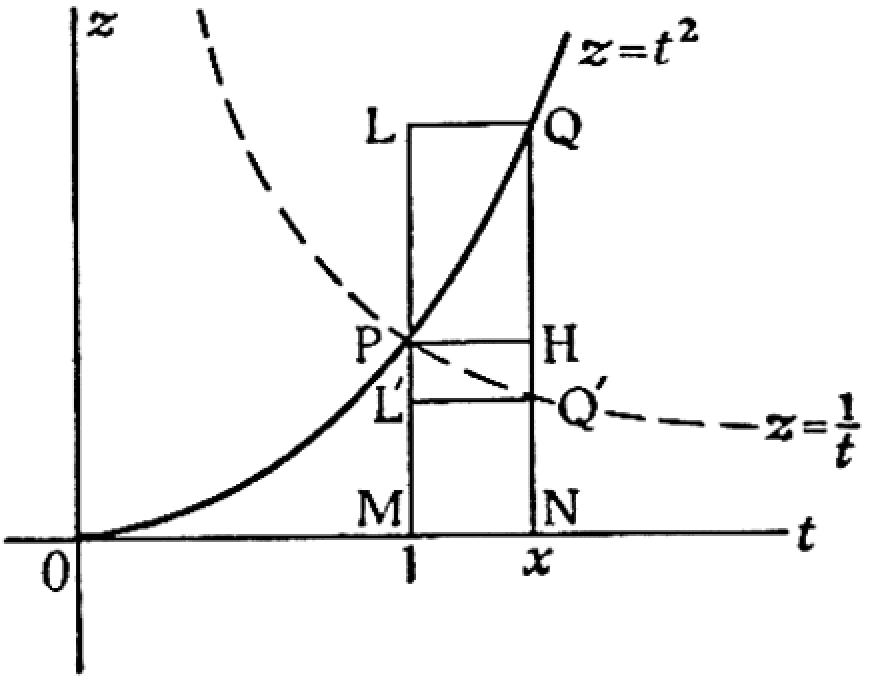

FIG. 53

Again, for all $m>0$, t^m is an increasing function of t, and so provided $b>a>0$ we have

$$a^m(b-a) < \int_a^b t^m\,dt < b^m(b-a) \quad \text{(see Fig. 54)},$$

$$\text{i.e. } a^m(b-a) < \frac{1}{m+1}(b^{m+1}-a^{m+1}) < b^m(b-a). \quad \ldots\ldots\ldots\ldots\text{(I)}$$

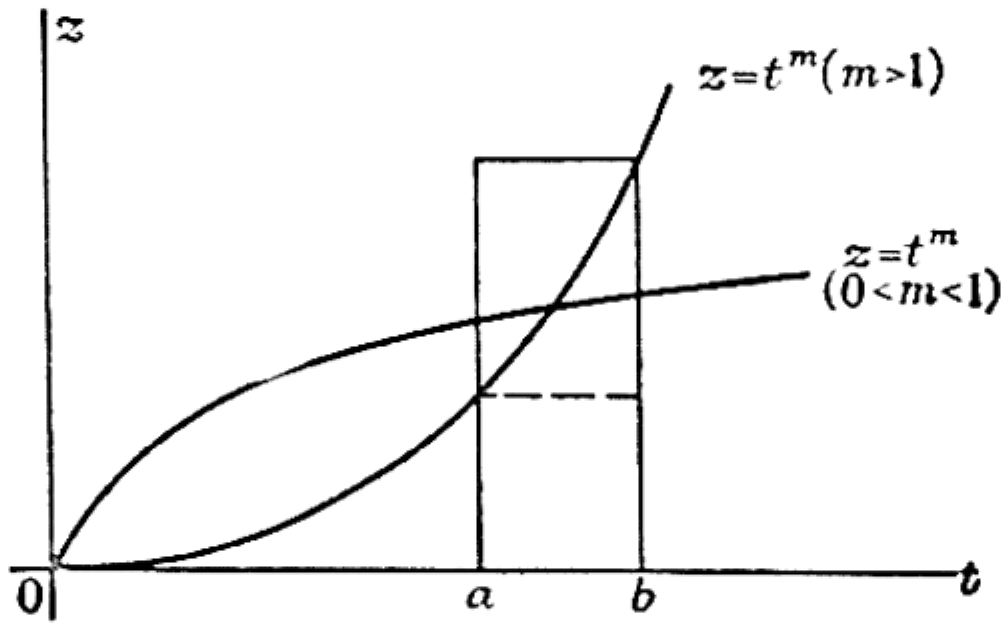

FIG. 54

Examples 123

1. By considering the function $z=t^{\frac{1}{2}}$ show that $x^{\frac{3}{2}}>3x^{\frac{1}{2}}-2$ if $x>1$.

2. If $0<x<1$, prove $1-x>-x\log x>x(1-x)$.

3. If $p>q>0$ and $t>1$, $t^p - t^q$ is positive; by integrating from 1 to x show that $\dfrac{x^{p+1}-1}{p+1} > \dfrac{x^{q+1}-1}{q+1}$.

Show the same inequality true if $0<x<1$.

4. If $m<0$ and $y>x>0$, show

$$x^m(y-x) > \frac{1}{m+1}(y^{m+1}-x^{m+1}) > y^m(y-x).$$

5. By replacing b by $(1+z)$ and a by 1 in (I) above, deduce

$$(1+z)^p > 1+pz \quad \text{if } p>1 \quad \text{and} \quad z>0.$$

6. By considering the graph of $z=e^t$ show $e^x>1+x$ and $e^x<\dfrac{1}{1-x}$ if $x<1$.

Use of a Graph

Some of the inequalities discussed on pp. 318–24 can be illustrated by the use of the graph of $y=x^k$ for positive x when $k>1$; or the graph may be used as an alternative proof.

If $k>1$ the curve $y=x^k$ is *convex* to the x-axis, which means that the mid-point of any chord is *above* the curve, and from which it follows that the centre of mass of equal particles at points of the curve will be above the curve.

Fig. 55 shows part of $y=x^3$.

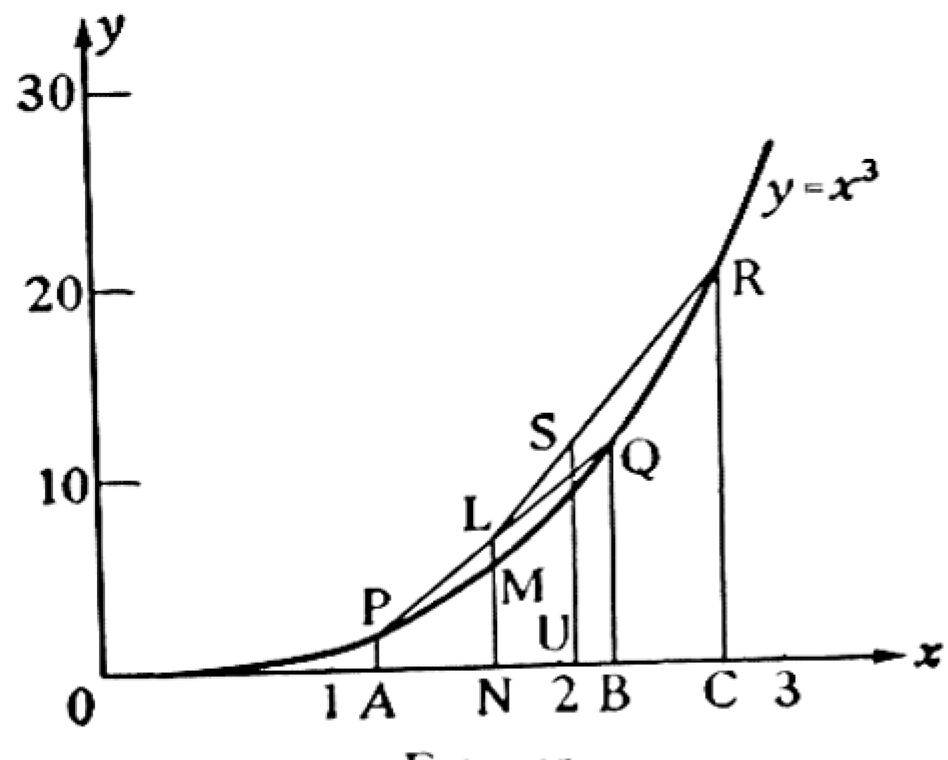

FIG. 55

Take points P, Q, R on it corresponding to values a, b, c for x ($OA=a$, etc.).

If L bisects PQ, $PA=a^3$, $QB=b^3$, so that $LN=\frac{1}{2}(a^3+b^3)$.

Also $ON=\frac{1}{2}(a+b)$, therefore $MN=\{\frac{1}{2}(a+b)\}^3$.

But L is above the curve, so that $LN > MN$.

$$\therefore \quad \frac{a^3+b^3}{2} > \left(\frac{a+b}{2}\right)^3.$$

Again, if S is taken on LR so that $LS = \frac{1}{2}SR$, S is the centre of mass of equal particles at P, Q, R, therefore $SU = \frac{1}{3}(a^3+b^3+c^3)$.

But $OU = \frac{1}{3}(a+b+c)$, and, $TU = \{\frac{1}{3}(a+b+c)\}^3$ T being the point where SU meets the curve.

Since $SU > TU$, $$\frac{a^3+b^3+c^3}{3} > \left(\frac{a+b+c}{3}\right)^3.$$

This process applies for any value of k if >1; k need not be an integer.

If $k = 2{\cdot}5$ we get $$\frac{a^{2\cdot5}+b^{2\cdot5}}{2} > \left(\frac{a+b}{2}\right)^{2\cdot5}$$

and $$\frac{a^{2\cdot5}+b^{2\cdot5}+c^{2\cdot5}}{3} > \left(\frac{a+b+c}{3}\right)^{2\cdot5}.$$

The process can be extended by taking four or more points on the curve.

Examples 124

1. Show that the method using the graph proves that

$$\frac{a^{3\cdot5}+b^{3\cdot5}+c^{3\cdot5}+d^{3\cdot5}}{4} > \left(\frac{a+b+c+d}{4}\right)^{3\cdot5} \text{ unless } a=b=c=d.$$

2. Use the graph of $y = x^{\frac{1}{2}}$ which is *concave* to the x-axis to show that if $a \neq b$

$$\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2} < \left(\frac{a+b}{2}\right)^{\frac{1}{2}}.$$

3. By using the graph of $y = e^x$, show that $e^p + e^{-p} > 2$ for all $p \neq 0$.

Miscellaneous Examples 125

1. For what values of x is $-1 < \dfrac{2x-3}{x+1} < 1$?

2. Show that both the quadratic expressions

$$2x^2 - 11x + 5 \quad \text{and} \quad 3x^2 - 7x - 6$$

will be negative provided $2x^2 - 7x + 3 < 0$.

3. If $y = \dfrac{3x^2+2x}{x^2+5}$, show that $y \leqslant 3\frac{1}{5}$ for all values of x.

4. For what values of x is $x^3 > 4x^2 - x - 6$?

5. Show that $x(x^2+x+1)$ is $>$ or <3 according as x is $>$ or <1.

6. If all the letters are positive, prove that

(i) $a^3+b^3 \geqslant a^2b+ab^2$;
(ii) $a^5+b^5 \geqslant a^3b^2+a^2b^3$;
(iii) $3(a^4+b^4+c^4) \geqslant (a^3+b^3+c^3)(a+b+c)$;
(iv) $(ab+bc+ca)^4 \geqslant 3abc(a+b+c)$.

7. If $a^2+b^2>c^2$, prove that $a+b>c$. Consider a triangle to show that the converse is not true.

8. If a, b, c are sides of a triangle and $2s=a+b+c$, prove that any two of $\sqrt{(s-a)a}$, $\sqrt{(s-b)b}$, $\sqrt{(s-c)c}$ are together greater than the third.

9. If any two of a, b, c are together greater than the third, prove that

(i) $a^2+b^2+c^2<2(ab+bc+ca)$;
(ii) $(b+c-a)(c+a-b)(a+b-c)<abc$;
(iii) $(b+c-a)^2+(c+a-b)^2+(a+b-c)^2>ab+bc+ca$.

10. If $p \neq q$ and $p+q \geqslant 2a$, and if m is a positive integer, prove that $p^m+q^m>2a^m$.

11. Show that $\sum_1^n r^2>n\left(\frac{n+1}{2}\right)^2$.

12. By considering the graph of $y=x^3$ for $x>1$ show that

(i) $3x^4>4x^3-1$; (ii) $x^4>4x-3$; (iii) $2(x^3+1)>(x+1)^3$.

13. Deduce from the graphs of $y=\frac{1}{x}$ and $y=\frac{1}{x^2}$ for positive x that

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>\frac{9}{a+b+c}$. (ii) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}>\frac{9}{a^2+b^2+c^2}$.

14. If $a>b>c$ show that $a^4b+b^4c+c^4a>ab^4+bc^4+ca^4$.

15. If m is a positive integer, show that $(m+1)^2<10m^2$, and use this result to prove that $m^2<10^m$.

Deduce that $\frac{m}{10^m}<\frac{1}{m}$ and that $\frac{m}{10^m}\to 0$ as $m\to\infty$; hence show that $\frac{\log n}{n}\to 0$ as $n\to\infty$.

16. If $0<x<1$ so that $x=10^{-k}$ where k is positive, show that $\log(n^r x^n)$ may be written as $-n\left(k-r\frac{\log n}{n}\right)$.

Deduce that $n^r x^n \to 0$ as $n\to\infty$.

CHAPTER XIV

DETERMINANTS

Three-row Determinants

In Chapter I the simplest case of the *determinant* notation was introduced, viz. the use of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ to mean $ad-bc$. This is what is called a 2×2 or two-row determinant.

The next determinant in order of difficulty, the 3×3 or 3-row determinant, is defined as follows; it is now convenient to use suffixes.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

In this expression the three letters in the first row, taken with alternate signs, are multiplied each by the two-row determinant obtained by omitting the row and column in which it lies.

Examples 126 [*All* the following should be worked]

1. Find the values of:

(i) $\begin{vmatrix} 2 & 1 & 7 \\ 3 & 3 & 5 \\ 0 & 1 & 6 \end{vmatrix}$; (ii) $\begin{vmatrix} 3 & 2 & 5 \\ 1 & 4 & 6 \\ 2 & 3 & 7 \end{vmatrix}$; (iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$;

(iv) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 1 & 1 \end{vmatrix}$; (v) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$; (vi) $\begin{vmatrix} 4 & 3 & 2 \\ 5 & 1 & 7 \\ 3 & 2 & 6 \end{vmatrix}$;

(vii) $\begin{vmatrix} 3 & 0 & 1 \\ 5 & 2 & 1 \\ 5 & 5 & 6 \end{vmatrix}$; (viii) $\begin{vmatrix} 3 & 2 & -4 \\ 0 & 4 & 7 \\ 0 & 1 & 5 \end{vmatrix}$; (ix) $\begin{vmatrix} 0 & 1 & 2 \\ 3 & 5 & 0 \\ 1 & 3 & 6 \end{vmatrix}$.

2. Solve the equation: $\begin{vmatrix} x & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 6 \end{vmatrix} = \begin{vmatrix} x & 0 & 1 \\ 0 & 4 & 4 \\ 3 & 4 & 6 \end{vmatrix}$.

3. Solve the equations: (ii) $\begin{vmatrix} x & 2x & 3x \\ 5 & 1 & 2 \\ 7 & 0 & 5 \end{vmatrix} + \begin{vmatrix} x & 0 \\ 0 & x \end{vmatrix} = 0$;

(ii) $\begin{vmatrix} (x+3) & 2x & (x+1) \\ (x-5) & -2 & (x-2) \\ 5 & 5 & 8 \end{vmatrix} = 0.$

4. Show that in the expansion of the general 3×3 determinant on p. 330, the second term may be written $+a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix}$.

5. Show that the 3-row determinant may be expanded with any row or column as multipliers; e.g. that it equals

$$-b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix},$$

or

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

[Notice that the + and − signs alternate; for a_1 +, for a_2 −, for b_2 +, etc.]

6. Show that rows and columns may be interchanged, i.e. that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

7. Show that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

[This determinant has great importance in connection with conics.]

8. Show that a 3×3 determinant with either two rows the same or two columns the same is identically zero.

[If the first part has been proved the second follows from No. 6.]

9. Show that if the letters in any one row (or column) are each multiplied by k, then the determinant is multiplied by k.

10. Show that $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

[See (ii) below for the general property.]

Properties of the Determinant

The results of the preceding examples may be summed up as follows:

(i) *Rows and columns may be interchanged.*

Hence any property true for rows will also be true for columns.

(ii) *If two adjacent rows (or columns) are interchanged, the determinant changes its sign, but not its numerical value.*

(iii) *If two rows (or columns) are the same, the determinant equals zero.*

(iii) follows from (ii), for if $\Delta = -\Delta$, then $\Delta = 0$.

Four-row determinants and others with more rows are defined in such a way that the properties (i), (ii), (iii) remain true.

Minors and Co-Factors

The minor determinant of a given letter or element is the determinant obtained by leaving out the row and column in which the letter occurs.

Thus in $\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ the minor of a_2 is $\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$.

and that of c_3 is $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

By definition :

$$\Delta = a_1 \times \text{minor of } a_1 - a_2 \times \text{minor of } a_2 + a_3 \times \text{minor of } a_3.$$

It is, however, more convenient to change the sign of the minor of a_2, calling the result the *co-factor* of a_2 or the *prepared minor* of a_2.

Denoting these co-factors by A_1, A_2, A_3, we have

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3$$

and similarly $\quad = b_1B_1 + b_2B_2 + b_3B_3$

or $\quad = a_1A_1 + b_1B_1 + c_1C_1.$

Notice particularly that $b_1A_1 + b_2A_2 + b_3A_3 = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$

Thus if the co-factors of the elements of one row are multiplied by the corresponding elements of another row the result is zero.

Four-row Determinants

Determinants of four or more rows are defined like those of three rows.

Thus
$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \equiv a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}$$
$$+ a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix},$$

each element of the first row being multiplied by the minor deter-

minant formed by leaving out the row and column in which that element lies and affixing signs, + and − alternately.

The properties stated on the previous page are true for determinants of four or more rows, but the proofs if worked out straightforwardly, as in Examples 126, Nos. 5, 6, 8, 9, 10, will be found long and troublesome. It will be best to assume them to be true without proof.*

Manipulation of Determinants

The properties of determinants which are of most value in changing them into simpler forms are :

1. *A determinant with two columns (or rows) identical vanishes.* This is because to interchange two columns changes the sign but not the numerical value of the determinant. A common factor of the terms in one column may be disregarded, since it merely multiplies the whole determinant given with the factor taken out.

2. $$\begin{vmatrix} x_1+x_2 & a_1 & a_2 \\ y_1+y_2 & b_1 & b_2 \\ z_1+z_2 & c_1 & c_2 \end{vmatrix} = \begin{vmatrix} x_1 & a_1 & a_2 \\ y_1 & b_1 & b_2 \\ z_1 & c_1 & c_2 \end{vmatrix} + \begin{vmatrix} x_2 & a_1 & a_2 \\ y_2 & b_1 & b_2 \\ z_2 & c_1 & c_2 \end{vmatrix}.$$

This follows from the way the determinant can be expanded by using the elements of the first column to multiply 2×2 determinants.

(The transformation above catches the eye better when shown with columns rather than with rows ; but it is equally true for rows, since rows and columns are interchangeable.)

3. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+kb_1+lc_1 & b_1 & c_1 \\ a_2+kb_2+lc_2 & b_2 & c_2 \\ a_3+kb_3+lc_3 & b_3 & c_3 \end{vmatrix}.$$

This is because, if the R.H.S. is separated as in 2 into three determinants, two of them will be zero.

The column altered need not be the first column.

Example I

Simplify and factorise the determinant Δ where

$$\Delta = \begin{vmatrix} a & a^2x - \frac{1}{x} & a^2x + \frac{1}{x} \\ b & b^2x - \frac{1}{x} & b^2x + \frac{1}{x} \\ c & c^2x - \frac{1}{x} & c^2x + \frac{1}{x} \end{vmatrix}.$$

* An alternative definition for determinants can be given, which can be shown to be equivalent to that used above, and from which the properties required can be proved without much difficulty. See No. 11, Exercise 128.

Add the 2nd column to the third to make a new second column.

$$\Delta = \begin{vmatrix} a & 2a^2x & a^2x+\frac{1}{x} \\ b & 2b^2x & b^2x+\frac{1}{x} \\ c & 2c^2x & c^2x+\frac{1}{x} \end{vmatrix} = \begin{vmatrix} a & 2a^2x & a^2x \\ b & 2b^2x & b^2x \\ c & 2c^2x & c^2x \end{vmatrix} + \begin{vmatrix} a & 2a^2x & \frac{1}{x} \\ b & 2b^2x & \frac{1}{x} \\ c & 2c^2x & \frac{1}{x} \end{vmatrix}.$$

The first of these vanishes, for (taking out the 2) the second and third columns are identical.

In the second determinant we can take out the common factors $2x$ and $\frac{1}{x}$ and get

$$\Delta = 2x \times \frac{1}{x} \times \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= 2[a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)]$$

$$= 2(b-c)(c-a)(a-b).$$

An alternative way of factorising the last determinant is shown in Example III.

Example II. Prove that

$$\Delta \equiv \begin{vmatrix} 1 & 2 & 4 & 6 \\ 2 & 3 & 5 & 7 \\ 3 & 4 & 6 & 8 \\ 4 & 5 & 7 & 9 \end{vmatrix} = 0.$$

Subtracting 3rd column from 4th and then 1st from second.

$$\Delta = \begin{vmatrix} 1 & 2 & 4 & 2 \\ 2 & 3 & 5 & 2 \\ 3 & 4 & 6 & 2 \\ 4 & 5 & 7 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 & 2 \\ 2 & 1 & 5 & 2 \\ 3 & 1 & 6 & 2 \\ 4 & 1 & 7 & 2 \end{vmatrix}.$$

Taking out the factor 2 common to the terms in 4th column, Δ has two columns identical and the result follows.

Factors

Suppose a determinant to involve a, b and c.

Then if it vanishes when a is replaced by b, it will have $a-b$ as a factor, by the factor theorem.

Symmetry may then show other factors.

Example III. Prove that each of the determinants

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a^2 & b^2 & c^2 \\ bc & ca & ab \\ 1 & 1 & 1 \end{vmatrix}$$

has $a-b$, $b-c$, $c-a$ as factors; and find the other factor in the second determinant.

If we put $a=b$ in either determinant the first two columns are identical, and so the determinant vanishes.

Hence in each case $a-b$ is a factor.

Similarly $b-c$ and $c-a$ are factors.

Considering in the first determinant the term a^2b, we see that the determinant $= -(a-b)(b-c)(c-a)$.

The second determinant has cyclic symmetry in a, b, c.

[It is not strictly symmetrical in a and b, since to interchange these letters changes the sign of the determinant. But the factor $a-b$ takes care of this change of sign.]

$\therefore$ the fourth factor is symmetrical and it is linear (since the whole is of the fourth degree);

$\therefore$ the fourth factor is $a+b+c$, perhaps multiplied by a constant.

Considering the term in a^3c, we see that

$$\text{the determinant} = (a-b)(b-c)(c-a)(a+b+c).$$

Examples 127

1. Evaluate the following determinants, making use of the property No. 3, p. 333, in order to reduce the arithmetic.

(i) $\begin{vmatrix} 17 & 7 & 5 \\ 47 & 21 & 16 \\ 29 & 14 & 11 \end{vmatrix}$; (ii) $\begin{vmatrix} 26 & 16 & 7 \\ 25 & 23 & 11 \\ 54 & 39 & 18 \end{vmatrix}$; (iii) $\begin{vmatrix} 35 & 36 & 23 \\ 20 & 25 & 16 \\ 38 & 39 & 25 \end{vmatrix}$;

(iv) $\begin{vmatrix} 33 & 31 & 22 \\ 12 & 11 & 8 \\ 19 & 17 & 12 \end{vmatrix}$; (v) $\begin{vmatrix} 58 & 36 & 15 \\ 28 & 12 & 9 \\ 48 & 23 & 19 \end{vmatrix}$; (vi) $\begin{vmatrix} 57 & 69 & 30 \\ 37 & 49 & 25 \\ 32 & 38 & 15 \end{vmatrix}$.

2. If no two of a, b and c are equal, find x from

$$\begin{vmatrix} 1 & bc+ax & a^2 \\ 1 & ca+bx & b^2 \\ 1 & ab+cx & c^2 \end{vmatrix} = 0.$$

3. Solve the equations:

(i) $\begin{vmatrix} x & 2 & -2 \\ 2 & x & -2 \\ -2 & 2 & x \end{vmatrix} = 0$;

(ii) $\begin{vmatrix} x+3 & 3x+3 & x+1 \\ x-5 & x-7 & x-2 \\ x & x+3 & x+6 \end{vmatrix} = 0$; (iii) $\begin{vmatrix} x+3 & 2x & 3x-1 \\ x+1 & 4x-1 & 2x-1 \\ x+4 & 2x+5 & 3x \end{vmatrix} = 0.$

4. Evaluate : (i) $\begin{vmatrix} 1 & 8 & 9 & 10 \\ -8 & 1 & 1 & 1 \\ -12 & 1 & 3 & 1 \\ -20 & 1 & 3 & 1 \end{vmatrix}$; (ii) $\begin{vmatrix} 3 & 7 & 13 & 21 \\ 1 & 8 & 27 & 64 \\ 1 & 4 & 9 & 16 \\ 1 & 2 & 3 & 4 \end{vmatrix}$.

5. Solve the equations: (i) $\begin{vmatrix} x & a & b \\ a & x & b \\ a & b & x \end{vmatrix} = 0$; (ii) $\begin{vmatrix} x-a & a & a+b \\ a & x & b \\ a & b & x \end{vmatrix} = 0$.

6. Prove that : $\begin{vmatrix} 1+t_1^2 & 2t_1 & 1-t_1^2 \\ 1+t_2^2 & 2t_2 & 1-t_2^2 \\ 1+t_3^2 & 2t_3 & 1-t_3^2 \end{vmatrix} = -4(t_2-t_3)(t_3-t_1)(t_1-t_2)$.

7. Show that $\begin{vmatrix} a^4 & a^3 & a \\ b^4 & b^3 & b \\ c^4 & c^3 & c \end{vmatrix} = -abc(b-c)(c-a)(a-b)(bc+ca+ab)$.

Also show that if $\begin{vmatrix} a^4-1 & a^3 & a \\ b^4-1 & b^3 & b \\ c^4-1 & c^3 & c \end{vmatrix} = 0$ and no two of a, b, c are equal, then $abc(bc+ca+ab) = a+b+c$.

8. Prove that

$$\begin{vmatrix} y^2z^2+x^2 & yz+x & 1 \\ z^2x^2+y^2 & zx+y & 1 \\ x^2y^2+z^2 & xy+z & 1 \end{vmatrix} = (y-z)(z-x)(x-y)(x-1)(y-1)(z-1).$$

[Hint. Divide into 4 determinants and take out a factor common to each.]

9. Solve the equation :

$$\begin{vmatrix} cx+a+b & ax+b+c & bx+c+a \\ bx+c & cx+a & ax+b \\ (a+b)x+c & (b+c)x+a & (c+a)x+b \end{vmatrix} = 0.$$

10. Prove that $\begin{vmatrix} 4 & x+1 & x+1 \\ x+1 & (x+2)^2 & 1 \\ x+1 & 1 & (x+2)^2 \end{vmatrix} = 2(x+1)(x+3)^3$.

11. Find all the values of x if $\begin{vmatrix} x & x^2 & 1+x^4 \\ a & a^2 & 1+a^4 \\ 1 & 1 & 2 \end{vmatrix} = 0$.

Elimination and the Solution of Equations using Determinants

It was shown in Chapter I that the solution of the equations

$$b_1x + b_2y + b_3 = 0,$$
$$c_1x + c_2y + c_3 = 0,$$

$$\frac{x}{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}.$$

or that if
$$b_1x+b_2y+b_3z=0,$$
$$c_1x+c_2y+c_3z=0,$$

then
$$\frac{x}{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}}=\frac{y}{\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix}}=\frac{z}{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}.$$

If therefore the three equations:
$$a_1x+a_2y+a_3z=0,$$
$$b_1x+b_2y+b_3z=0,$$
$$c_1x+c_2y+c_3z=0,$$

are satisfied by the same values of $x:y:z$, then by substituting in the first equation the ratios given by the other two we get
$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}+a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix}+a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}=0,$$

or
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

This is the result of eliminating $x:y:z$ from the 3 equations and is called the *eliminant*. It is here that we see the beauty of the determinant notation; to get the relation between the a's, b's, c's we merely form the determinant with the coefficients.

In coordinate geometry we can take advantage of this in order to show lines concurrent or points collinear.

The three lines $a_1x+a_2y+a_3=0$, $b_1x+b_2y+b_3=0$, $c_1x+c_2y+c_3=0$ are concurrent if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$

Again, the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear if l, m, n can be found so that
$$lx_1+my_1+n=0, \quad lx_2+my_2+n=0, \quad lx_3+my_3+n=0,$$

and the condition, by eliminating $l:m:n$ is $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$

Now suppose we are asked to solve the equations:
$$a_1x+a_2y+a_3z+a_4=0,$$
$$b_1x+b_2y+b_3z+b_4=0,$$
$$c_1x+c_2y+c_3z+c_4=0.$$

Using the notation of co-factors in the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \Delta,$$

as described on p. 332, if we multiply the 3 equations by A_1, B_1, C_1 respectively and add, we get

$$x(a_1A_1+b_1B_1+c_1C_1)+y(a_2A_1+b_2B_1+c_2C_1)+z(a_3A_1+b_3B_1+c_3C_1) \\ +(a_4A_1+b_4B_1+c_4C_1)=0.$$

The coefficients of x, y, z and the constant term are

$$\Delta, 0, 0, \begin{vmatrix} a_4 & a_2 & a_3 \\ b_4 & b_2 & b_3 \\ c_4 & c_2 & c_3 \end{vmatrix} \quad \text{and so} \quad x=\begin{vmatrix} a_4 & a_2 & a_3 \\ b_4 & b_2 & b_3 \\ c_4 & c_2 & c_3 \end{vmatrix} \div(-\Delta).$$

Similarly y can be found by multiplying the equations by A_2, B_2, C_2 to give $\Delta \,.\, y+(a_4A_2+b_4B_2+c_4C_2)=0$; i.e.

$$y=\begin{vmatrix} a_1 & a_4 & a_3 \\ b_1 & b_4 & b_3 \\ c_1 & c_4 & c_3 \end{vmatrix} \div(-\Delta),$$

and in the same way $z=\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} \div(-\Delta).$

If a fourth equation $d_1x+d_2y+d_3z+d_4=0$ is satisfied by the same values of x, y, z, then

$$d_1\begin{vmatrix} a_4 & a_2 & a_3 \\ b_4 & b_2 & b_3 \\ c_4 & c_2 & c_3 \end{vmatrix}+d_2\begin{vmatrix} a_1 & a_4 & a_3 \\ b_1 & b_4 & b_3 \\ c_1 & c_4 & c_3 \end{vmatrix}+d_3\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}-d_4\Delta=0. \quad ....(\alpha)$$

This is the eliminant of x, y, z between the four equations and is equivalent to expanding the 4×4 determinant, making use of the property about interchanging columns:

$$\begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}=0. \quad ....(\beta)$$

An easy way of writing down the solution of the three given equations in x, y, z is to remember this result of eliminating with a fourth equation

$$d_1x+d_2y+d_3z+d_4=0.$$

The eliminant is the determinant (β) above, and so is expanded into (α) above.

If (α) is regarded as being the same as $d_1x+d_2y+d_3z+d_4=0$, we see that the values of x, y, and z are $\begin{vmatrix} a_4 & a_2 & a_3 \\ b_4 & b_2 & b_3 \\ c_4 & c_2 & c_3 \end{vmatrix} \div(-\Delta)$, etc.

Example I. Find x from the equations:

$$\begin{aligned} 3x+2y+4z+\ \ 7&=0, \\ 2x-3y+5z-\ \ 8&=0, \\ 5x-7y-2z+10&=0. \end{aligned}$$

Proceeding as in the general case we have

$$x\begin{vmatrix} 3 & 2 & 4 \\ 2 & -3 & 5 \\ 5 & -7 & -2 \end{vmatrix} + \begin{vmatrix} 7 & 2 & 4 \\ -8 & -3 & 5 \\ 10 & -7 & -2 \end{vmatrix} = 0.$$

The first determinant $=3(6+35)+2(25+4)+4(-14+15)$
$=123+58+4=185.$

The second determinant $=7\times 41+2\times 34+4\times 86.$
$=287+68+344=699.$

The value of x, therefore, is $-699/185$.

Example II. Eliminate $x:y:z$ from the equations:

$$\begin{aligned} ax+by+cz&=0, \\ bz+cy+az&=0, \\ cx+ay+bz&=0, \end{aligned}$$

and simplify the result.

The eliminant is $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \equiv \Delta = 0.$

[Add all columns for new 1st column and take out the common factor $a+b+c$.]

$$\Delta=(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

[expanding from 1st column]

$$\begin{aligned} \Delta &= (a+b+c)\{cb-a^2+ac-b^2+ab-c^2\} \\ &= -(a^3+b^3+c^3-3abc). \end{aligned}$$

The eliminant is $a^3+b^3+c^3=3abc$.

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1. Solve the equations: $\begin{aligned} 2x+3y+4z&=6, \\ 4x+9y+16z&=36, \\ 8x+27y+64z&=216. \end{aligned}$

2. Find a if the three equations

$$\begin{aligned} 2x+3y-5z&=0,\\ x+2y+4z&=0,\\ ax-5y+3z&=0, \end{aligned}$$

are satisfied by the same value of $x:y:z$.

Repeat this, making the first equation

$$2x+3y+5z=0.$$

3. Solve for x if

$$\begin{aligned} x+y+z&=1,\\ ax+by+cz&=1,\\ a^2x+b^2y+c^2z&=1. \end{aligned}$$

4. If

$$\begin{aligned} (a+b)^2x+bcy+ac&=0,\\ bcx+(a+c)^2y&=0, \end{aligned}$$

and

$$acx+aby+(b+c)^2=0,$$

and if no one of the letters a, b, c is zero, prove that

$$a+b+c=0.$$

5. Find the value of $\begin{vmatrix} 47 & 98 & 109 \\ 41 & 92 & 104 \\ 46 & 96 & 105 \end{vmatrix}$.

6. Prove that

$$\begin{vmatrix} 0 & x & y & 0 \\ x & 0 & 0 & a \\ y & 0 & 0 & b \\ 0 & a & b & 0 \end{vmatrix} = \begin{vmatrix} x & y \\ a & b \end{vmatrix}^2.$$

7. Show that $\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}$ vanishes if $x=y+z$.

Use symmetry to show that the determinant

$$=(x+y+z)(x-y-z)(y-z-x)(z-x-y).$$

8. Being given that the condition that

$$ax^2+2hxy+by^2+2gx+2fy+c$$

should have linear factors is $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$ and $ab-h^2<0$,

show that two of the following have linear factors but that the other has not:

(i) $2x^2-2xy-4y^2+5x+8y-3$;

(ii) $4x^2+2xy-y^2-4x+4y-3$;

(iii) $4x^2+11xy+6y^2+21x+17y+5$.

9. If
$$\begin{aligned} x+2y+\ \ z&=8,\\ x+\ \ y-\ \ z&=0,\\ 2x-2y+3z&=7, \end{aligned}$$
show that $y=2$ and $x-2y+z=0$.

10. Eliminate $x:y:z:\lambda$ between the equations:
$$\begin{aligned} ax+hy+gz&=\lambda l,\\ hx+by+fz&=\lambda m,\\ gx+fy+cz&=\lambda n,\\ lx+my+nz&=0, \end{aligned}$$
giving the eliminant in determinant form, and also in the form obtained by solving for x, y, z from the first 3 equations and substituting in the fourth.

11. Complete the statement $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + \dots$.

Can you see any rule about the *order* of the suffixes 1, 2, 3 in the six terms which would settle the *signs* of the terms?

12. Show that the determinant
$$\begin{vmatrix} x & y & z & x+y+z \\ y & x & x+y+z & z \\ z & x+y+z & x & y \\ x+y+z & z & y & x \end{vmatrix}$$
contains x as a factor; and evaluate the determinant. (N.U.J.B.)

13. Show that $\begin{vmatrix} 0 & 1^2 & 2^2 & 3^2 \\ 1^2 & 0 & 1^2 & 2^2 \\ 2^2 & 1^2 & 0 & 1^2 \\ 3^2 & 2^2 & 1^2 & 0 \end{vmatrix} = 0.$

CHAPTER XV

THEORY OF NUMBERS

Primes and Composite Numbers

This chapter deals with some properties of integers, and any letter which stands for a number stands for a positive integer unless the contrary is stated.

A *prime* number has no factors, not counting itself and unity, and one number is said to be prime to another when the two numbers have no common factor other than unity ; for instance, 13 is a prime number, and 6 is prime to 35 although neither 6 nor 35 is prime, each being a *composite* number.

Prime numbers may be found by writing down the integers 2, 3, 4, 5, 6, 7 ... as far as desired and crossing out first of all the multiples of 2, then all the multiples of 3, then all the multiples of the first number which is not crossed out (5), and continuing this process.* The numbers not crossed out are the primes.

Since every composite number has a factor not greater than its square root, crossing out multiples of primes up to 7 will show all primes up to 121, while going as far as 13 will show all primes up to 289.

Various properties follow from such facts as that one of a consecutive pair of numbers is even or again that one of three consecutive numbers is a multiple of 3.

Example I. The product of two consecutive even numbers is divisible by 8.

Every second even number is a multiple of four, so of the two even numbers one is a multiple of 4 while the other, being even, is a multiple of 2. The product therefore is a multiple of 8.

Example II. The difference between a number and its cube is divisible by 6.

If n is the number, the difference is $n^3 - n$,

$$\text{i.e. } n(n^2 - 1) \equiv (n-1)n(n+1),$$

$n - 1$, n and $n + 1$ are three consecutive numbers of which one must be a multiple of 3 and at least one must be even.

$$\therefore\ n^3 - n \text{ is divisible by } 6.$$

* In doing this, we are using the " *Sieve of Eratosthenes* ".

Example III. Show that no square is of either of the forms $7m-1$ or $7m-2$.

Every number is of one of the forms $7n$, $7n \pm 1$, $7n \pm 2$, $7n \pm 3$.

The corresponding squares are of the forms

$$7m,\ 7m+1,\ 7m+4,\quad 7m+9=7(m+1)+2,$$

none of which is of the form $7m-1$ or $7m-2$. Q.E.D.

Examples 129

1. Show that the product of 3 consecutive positive integers of which the middle one is odd must be divisible by 24.
2. Show that the difference between the squares of any two odd numbers is divisible by 8.
3. For all n show $n(n+1)(2n+1)$ to be a multiple of 6.
4. Show that the sum of the squares of 3 consecutive odd numbers is one less than a multiple of 12.
5. If n is odd, prove $(n^2+3)(n^2+15)$ divisible by 32.
6. The fifth pair of primes which differ by 2 (starting with 3 and 5) is the pair 29 and 31. Find three more pairs of primes which differ by 2 between 35 and 75.
7. By considering the product $(p-1)(p)(p+1)$, show that if $p(>3)$ is prime, then p^2-1 is divisible by 24.

   Deduce that the difference of the squares of two prime numbers each greater than 3 is divisible by 24; also that either the sum or the difference of two prime numbers each greater than 3 is a multiple of 3.
8. Show that the square of any number has one of the forms $5k$, $5k \pm 1$.
9. Show that every number which is a square must be of the form $4k$ or $4k+1$.

   Deduce that the sum of two odd numbers each of which is a square cannot itself be a square.
10. If p is a prime greater than 2 show $9p^2$ is of the form $8k+1$.
11. If n is even show that $n(n+2)(n+10)$ is either a multiple of 96 or of 48.

Prime Factors

The prime factors of a number can be found by dividing it repeatedly by 2 until the quotient is odd; dividing this quotient by 3 as long as 3 goes in evenly, and carrying on this process with 5, 7, 11, 13, ... until the last prime divisor is as big as the square root of the quotient.

Thus to find the prime factors of 1564068 :

```
 2 ) 1564068                                              Factors
    ‾‾‾‾‾‾‾‾
  2 ) 782034                                                $2^2$
     ‾‾‾‾‾‾‾
  3 ) 391017                                                 3
     ‾‾‾‾‾‾‾
 11 ) 130339          [5, 7 not factors]                    11
     ‾‾‾‾‾‾‾
  13 ) 11849
      ‾‾‾‾‾‾
         911 ... 6      [13 not a factor]
 17 ) 11849 | 697
     ‾‾‾‾‾‾‾‾‾‾‾‾
      102
      ‾‾‾
       164          17 ) 697 | 41
       153                68                                $17^2$
       ‾‾‾                ‾‾
        119                17
        119                17                               41
        ‾‾‾                ‾‾
```

Thus $1564068 = 2^2 . 3 . 11 . 17^2 . 41$.

The above process makes it seem clear that the result is unique. This is the case.

It is not possible for a number to be expressed as the product of prime factors in more than one way.

If it were and there were different powers of the same primes in the two ways we should have a result like $2^5 . 3^2 = 2^2 . 3^4$, which would reduce to $2^3 = 3^2$ which (if true, which it isn't) would show that 2 and 3 were not prime to each other.

If, however, there were powers of different primes in the two ways, we should have a result like $2^5 . 3^2 = 2 . 3 . 7^2$, which would reduce to $2^4 . 3 = 7^2$ and would show that either 2 or 3 is not prime to 7.

This argument can be applied generally by supposing that a number N has been expressed as the product of prime factors in two ways; then removing the common factors, if any, from the two supposed equal expressions we should have

$$A^a B^b C^c \dots = P^p Q^q R^r \dots$$

where $A, B, C \dots, P, Q, R, \dots$, are different prime numbers.

But in order that the two expressions might be equal, the R.H.S. must be divisible by A, which is contrary to the assumption that the R.H.S. is made up of primes different from A.

H.C.F.

If two or more numbers have been expressed in prime factors their highest common factor (H.C.F.) can be written down at once.

Sometimes it is required to find the H.C.F. of two large numbers, or to show that one of them is prime to the other (their H.C.F. $=1$) and the division method now shown is preferable; it depends on the fact that if M and N have a common factor x, and if $M \div N$ gives remainder R, then x is a factor of R.

For let $M \equiv xy$ and $N \equiv xz$, M and N having common factor x.

Then $R \equiv xy - kxz$ for some integral k

$\equiv x(y-kz)$, i.e. R has x as a factor.

The division method of finding an H.C.F. is illustrated by two examples worked below; the series of divisions shown under (A) are arranged more compactly as in (B).

(i) Find the H.C.F. of 1073 and 851.

(A)
$$\begin{array}{rr|r|r|r|r|l}
851 & 1073 & 1 & & & & \\
 & 851 & & & & & \\ \hline
 & 222 & 851 & 3 & & & \\
 & & 666 & & & & \\ \hline
 & & 185 & 222 & 1 & & \\
 & & & 185 & & & \\ \hline
37 = \text{H.C.F.} & & & 37 & 185 & 5 & \\
 & & & & 185 & &
\end{array}$$

(B)
$$\begin{array}{r|r|r|l}
3 & 851 & 1073 & 1 \\
 & 666 & 851 & \\ \hline
5 & 185 & 222 & 1 \\
 & 185 & 185 & \\ \hline
 & & 37 = \text{H.C.F.} &
\end{array}$$

(ii) Find the H.C.F. of 2717 and 901.

$$\begin{array}{r|r|r|l}
64 & 901 & 2717 & 3 \\
 & 896 & 2703 & \\ \hline
1 & 5 & 14 & 2 \\
 & 4 & 10 & \\ \hline
\text{H.C.F.} = 1 & & 4 & 4 \\
 & & 4 &
\end{array}$$

As the H.C.F. $=1$, 901 is prime to 2717.

The Number of Factors

Each term in the product

$$(1+a+a^2+a^3+\ldots+a^p)(1+b+b^2+b^3+\ldots+b^q)$$
$$(1+c+c^2+c^3+\ldots+c^r) \quad \ldots\ldots\ldots\ldots(\alpha)$$

is a factor of $a^p b^q c^r$, and the number of such terms is

$$(p+1)(q+1)(r+1);$$

$\therefore$ the number of factors of $a^p b^q c^r$ (including unity) is

$$(p+1)(q+1)(r+1).$$

To find the number of ways in which $N \equiv a^p b^q c^r$ can be expressed as the product of two numbers, two cases must be considered.

(i) N not a square. In this case one of the numbers p, q, r must be odd, and so one of the numbers $(p+1)$, $(q+1)$, $(r+1)$ must be even; hence the number of factors of N is even and as they must be paired off to give the product N, this can be done in $\frac{1}{2}(p+1)(q+1)(r+1)$ ways.

(ii) N a square. $\sqrt{N}$ will be one of the $(p+1)(q+1)(r+1)$ factors of N and must be repeated to give $\sqrt{N} \times \sqrt{N}$.

Consequently the number of factors will be

$$\tfrac{1}{2}\{(p+1)(q+1)(r+1)+1\}.$$

Another deduction from the product (α) is that the sum of the factors of $a^p \,.\, b^q \,.\, c^r$ is $\dfrac{a^{p+1}-1}{a-1} \times \dfrac{b^{q+1}-1}{b-1} \times \dfrac{c^{r+1}-1}{c-1}$.

If the number of ways in which $N \equiv a^p b^q c^r d^s \ldots$ can be expressed as the product of two factors each prime to the other is needed, then the factors in a^p or b^q or c^r, etc., cannot be shared by the two factors.

$\therefore$ the number of ways required is equal to the number of ways of resolving $a \,.\, b \,.\, c \,.\, d \ldots$ into 2 factors,

$$\text{i.e. } \tfrac{1}{2}(1+1)(1+1)(1+1)\ldots \equiv 2^{n-1}$$

if n is the number of primes $a, b, c, d \ldots$.

Examples 130

1. Put into prime factors:

   (i) 4275; (ii) 2268; (iii) 13475.

2. Write down the H.C.F. of each pair of the numbers in No. 1.
3. Write down in factors the lowest common multiple (L.C.M.) of 4275 and 13475.
4. Find the H.C.F. of the following pairs of numbers:

   (i) 2160, 8505; (ii) 72880, 92011.

5. Show that the following pairs of numbers are prime to each other:

   (i) 2337, 2537; (ii) 24487, 116448.

6. For each of the numbers in Example 4 (i) find the number and sum of their different factors; also find the number of ways in which each can be expressed as the product of two factors, and how many of these pairs of factors are prime to each other.
7. Show that the sums of the divisors of 140 and of 164 are equal (the numbers themselves included).

8. A *perfect* number is one equal to the sum of its divisors including 1 but excluding itself (e.g. 6 is a perfect number; $6 = 1 + 2 + 3$).

Show 28, 496, 8128 are perfect numbers.

9. Show that if $2^n - 1$ is a prime number, then $2^{n-1}(2^n - 1)$ is equal to the sum of its divisors (itself excluded).

10. Write down the prime factors of the number $N = 2^6(2^7 - 1)$. Write down also, in index form, all numbers which divide exactly into N, including 1 but not N itself, and show that their sum is N.

11. Show that 720 is the least number that has 30 divisors (itself included).

12. Given that $11111 = 41 \times 271$, show that the multiplications $9 \times 41 = 369$ and $9 \times 271 = 2439$ give the decimals for $\frac{1}{271}$ and $\frac{1}{41}$, viz. $\cdot\dot{0}036\dot{9}$ and $\cdot\dot{0}243\dot{9}$. Treat 111 and 1111 in the same way; also 1,111,111 which has 239 as a factor.

[Hint. Multiply both sides of the given equality by 9.]

Notation

Two useful notations will now be introduced:

(i) Instead of writing "a is a multiple of b", we can write $a = M(b)$ where the M stands for "multiple of".

Note that if $a = M(b)$ and $c = M(b)$ it follows that

$$(a - c) = M(b),$$

not that a is necessarily equal to c.

$$\left.\begin{array}{l}\text{Again, if } \left.\begin{array}{l} a_1 = M(b) + r_1 \\ \text{and } a_2 = M(b) + r_2 \end{array}\right\} \text{then } a_1 a_2 = M(b) + r_1 r_2, \\ \text{and a similar result is true if there are } n \\ \text{factors.}\end{array}\right\} \quad \cdots\cdots\cdots(A)$$

(ii) If a and b leave the same remainder when divided by p we say a and b are *congruent* with p as *modulus*, and write this

$$a \equiv b \pmod{p},$$

or since in this case $a - b$ must be $M(p)$, we can write $a - b \equiv 0 \pmod{p}$.

Such statements are called *congruences*.

The student should note that here is an entirely new use of the words *congruent*, *congruence*, and *modulus*.

Theorem on Remainders

If a is prime to p and r and s numbers less than p, then ra and sa can *not* leave the same remainder when divided by p.

Or, using the above notation, $ra - sa \equiv 0 \pmod{p}$ is *not* possible.

For if $ra - sa = M(p)$, then since p is prime to a it follows that $r - s = M(p)$, which is not possible since r and s are both less than p.

From this it follows that $a, 2a, 3a, \dots, (p-1)a$ all give different remainders on division by p, and as the only possible remainders are $1, 2, 3, \dots, (p-1)$, the remainders must be these numbers in some order, usually *not* the order $1, 2, 3 \dots$.

Using the result (A) above, it follows that the product

$$a \cdot 2a \cdot 3a \dots (p-1)a = M(p) + 1 \cdot 2 \cdot 3 \dots (p-1)$$

or $(a^{p-1} - 1)(p-1)! = M(p)$, provided a is prime to p. ....(B)

Prime Numbers and Factorials

If p is prime, then it is not a factor of $(p-1)!$ since it is prime to each factor in $(p-1)!$.

On the other hand, if q is a composite number, then, *with one exception*, $(q-1)!$ is a multiple of q.

Suppose $q = a^r \cdot b^s \cdot c^t \dots$ where $a, b, c, \dots$ are primes.

If $r > 2$ it follows that $a^{r-1} < q$, and so both a and a^{r-1} are factors of $(q-1)!$.

If $r = 2$, both a and $2a$, unless $a = 2$ and there are no other factors, are less than q, and so a^2 is a factor of $(q-1)!$.

For example, if $q = 24 = 2^3 \cdot 3$, then $23!$ includes the factors $2, 2^2, 3$ and so $23! = M(24)$.

The exception is $q = 4 = 2^2$, for in this case $3! = 6$, which is not $M(4)$.

That $(p-1)!$ is not $M(p)$ if p is prime provides a test for primes which, however, is worthless since the labour of working out $(p-1)!$ and then dividing by p is so lengthy if p is large.

Fermat's Theorem

In the result (B) above, suppose p to be a prime number.

Then since $(p-1)!$ is prime to p, it follows that

$$a^{p-1} - 1 = M(p) \quad \text{if } a \text{ is prime to } p, \quad \text{i.e. if } a \neq M(p).$$

This is Fermat's Theorem.

If p is prime, either $a = M(p)$ or $a^{p-1} - 1 = M(p)$. Consequently, in all cases, $a^p - a = M(p)$.

There is no Limit to the Number of Primes

However large the prime p is, the product of the primes up to p, viz. $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \dots p$ is divisible by each of these prime numbers.

$\therefore\ 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \ldots p+1$ is a number which has none of the factors 2, 3, 5, 7, 11, ..., p, and consequently is either a prime or the product of prime numbers each greater than p. Therefore in either case there is a prime greater than p, and the theorem is established.

Example I. If a is prime to 21, prove that $a^6 - 1 = M(21)$.

By Fermat's Theorem $a^6 - 1 = M(7)$ for a being prime to 21 is prime to 7.

Also $a^2 - 1$ is a factor of $a^6 - 1$ and a is prime to 3.

$\therefore$ by Fermat's Theorem $a^2 - 1 = M(3)$.

The result follows.

Example II. If a is prime to 255, prove that $a^{16} - 1 = M(255)$.

Since a is prime to 255, it is prime to 17, 5 and 3, which are the factors of 255.

Hence, using Fermat's Theorem,

$$a^{16} - 1 = M(17);$$

also $a^4 - 1$ and $a^2 - 1$, which are factors of $a^{16} - 1$, are multiples of 5 and of 3 respectively.

The result follows.

Examples 131

1. Find x, y, z from $87 \equiv M(13) + x$, $99 = M(7) + y$, $115 = M(31) + z$.
2. What is x if $\{M(p) + 1\}\{M(p) - 2\}\{M(p) - 7\} = M(p) + x$?

   [Nos. 3 to 7 illustrate that congruences behave like equations as regards addition, subtraction and multiplication, but that as regards division a modification is needed.]
3. Prove that if $a_1 \equiv b_1 \pmod p$ and $a_2 \equiv b_2 \pmod p$, then
   $a_1 + a_2 \equiv b_1 + b_2$, $a_1 - a_2 \equiv b_1 - b_2$ and $a_1 a_2 \equiv b_1 b_2 \pmod p$.
4. Show that $47 \times 5 \equiv 3 \times 5 \pmod{10}$, but that 47 is not congruent to $3 \pmod{10}$.

   [This is because 5 is a factor of 10; it can be inferred that
   $$47 \equiv 3 \pmod 2.]$$
5. Show that if $xa \equiv ya \pmod{za}$, then $x \equiv y \pmod z$.
6. Show that $51 \times 6 \equiv 9 \times 5 \pmod 7$ and that $51 \equiv 9 \pmod 7$.
7. Show that if $xa \equiv ya \pmod z$ and if z is prime to a, then
   $$x \equiv y \pmod z.$$
8. Show that when n is even, $n^5 - n$ is divisible by 30, and when n is odd it is divisible by 120.
9. Considering the various factors of the form $x^s - 1$ possessed by $x^{25} - x$, show that it is a multiple of $3 \cdot 5 \cdot 7 \cdot 13$ if x is prime to each of these numbers.

10. (i) Use Fermat's Theorem to prove that $a^{12}-1$ is a multiple of 1365 provided a is prime to this number.
(ii) Further, if a is itself prime, prove that $a^{12}-1=M(21840)$. [$(a^3-1)(a^3+1)(a^6+1)$ is $M(16)$.]
(iii) Show also, since a is prime to 3 and so $a^2=M(3)+1$, that $a^6-1=M(9)$, and hence that $a^{12}-1=M(65520)$.

12. Prove, if p is prime and n prime to it, that $n^{(p-1)/2}$ is either $M(p)+1$ or $M(p)-1$.

13. If x is a prime greater than 5, prove $x^8-1=M(480)$.

14. If x is a prime greater than 7, prove $x^6-1=M(504)$.

15. Show that no number of the form n^4+4 other than 5 can be prime.

16. If a and b are unequal and neither is $M(19)$, prove that either a^9-b^9 or a^9+b^9 is $M(19)$.

17. If a and b are both prime to 255, show that $a^{16}-b^{16}$ is divisible by 255.

18. Obtain further primes from

$$2\cdot3\cdot5+1,\ 2\cdot3\cdot5\cdot7+1,\quad 2\cdot3\cdot5\cdot7\cdot11+1.$$

19. Show by multiplication and division that

$$6!+1=M(7)\ \text{and}\ 10!+1=M(11).$$

20. Find the prime factors of $21!$.
[Dividing 21 repeatedly by 2 gives quotients 10, 5, 2, 1. These are the numbers of times that multiples of 2, 2^2, 2^3, 2^4 occur in $21!$.
So the power of 2 in $21!$ is $2^{10+5+2+1}$, i.e. 2^{18}.
Similarly for successive divisions by 3, 5, 7, 11, 13, 17, 19, the last four of these occurring only once].

21. Find the prime factors of $52!$.

The Product of Consecutive Integers

In the expansion of $(1+x)^n$, if n is an integer, each coefficient is integral and the coefficient of x^r is $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$.

Thus *the product of r consecutive integers is divisible by r!.*

If p is prime and n is prime to p and greater than p, since $n(n-1)(n-2)\ldots(n-p+1)$ is a multiple of $p!$, it is a multiple of p.

Also as n is prime to p,

$\therefore\ (n-1)(n-2)\ldots(n-p+1)$ is a multiple of p,

i.e. $(n-1)(n-2)\ldots(n-p+1)\equiv 0 \pmod{p}$. ....(i)

$\therefore\ (n-1)(n-2)\ldots(n-p+1)\equiv n^{p-1}-1 \pmod{p}$ ....(ii)

by Fermat's Theorem.

This is also true if $n<p$, for then one of the factors in the product on the L.H.S. is 0.

(i) is not true if $n=p$ or $M(p)$, but it will be shown later that (ii) is true for $n=p$.

Notice also that if n is prime, ${}^nC_r$ is a multiple of n, since in this case none of the factors in $r!$ can divide n.

Wilson's Theorem. $(p-1)!+1=M(p)$ if p is prime

Take first the example when $p=13$ and consider the numbers 1 to 12.

The numbers from 2 to 11 can be paired together thus:

$$2\times 7,\ 3\times 9,\ 4\times 10,\ 5\times 8,\ 6\times 11,$$

so that in each case the product is $M(13)+1$.

Also $1\times 12=M(13)-1$.

Hence $\quad 12!=\{M(13)-1\}\{M(13)+1\}=M(13)-1$

$\quad$ or $12!+1=M(13)$.

Now take the general case when the prime is p and consider the numbers 1 to $(p-1)$.

If s is one of these, one and only one of the products

$$1\,.\,s,\ 2\,.\,s,\ 3\,.\,s,\ 4\,.\,s,\ \ldots,\ (p-1)\,.\,s \text{ is } M(p)+1.$$

Suppose this one to be $r\,.\,s$.

If $r=s$ then $s^2-1=M(p)$ or $(s+1)(s-1)=M(p)$.

$\therefore$ either $s+1$ or $s-1$ is $M(p)$; $\quad \therefore\ s=p-1$ or 1 since $s<p$.

$\therefore$ the numbers from 2 to $p-2$ (there are $p-3$ of them, an even number) can be paired together so that the product of each pair is $M(p)+1$; also $1\,.\,(p-1)=M(p)-1$;

$$\therefore\ (p-1)!=\{M(p)+1\}\{M(p)-1\}=M(p)-1,$$

and Wilson's Theorem is proved.

$(n-1)(n-2)\ldots(n-p+1)-n^{p-1}+1\equiv 0 \pmod p$ if p is prime

This has been proved if $n>p$ and prime to it, and also if n is one of the numbers $1, 2, 3, \ldots, (p-1)$ by Fermat's Theorem.

If $n=0$ or $n=p$ it is true by Wilson's Theorem (since in the latter case p^{p-1} is itself $M(p)$).

The theorem is also true if

$$n=kp,$$

for $kp-1,\ kp-2,\ \ldots,\ kp-(p-1)$ are congruent to $-1,\ -2,\ \ldots,$

$-(p-1)$, and the product of these is $(p-1)!$ since their number is even.

The theorem is therefore true for all positive values of n and for $n=0$.

Also, since it follows for n if it is true for $kp+n$, the thəorem is true for negative values of n.

Lagrange's Theorem

The theorem

$$(n-1)(n-2)\ldots(n-p+1)-n^{p-1}+1\equiv 0 \pmod p \text{ if } p \text{ is prime}$$

can be written

$$-n^{p-2}\Sigma s_1+n^{p-3}\Sigma s_1s_2-n^{p-4}\Sigma s_1s_2s_3+\ldots+(p-1)!+1\equiv 0 \pmod p,$$

when $s_1, s_2, \ldots$ are the numbers $1, 2, 3, \ldots, (p-1)$.

But by Wilson's Theorem $(p-1)!+1\equiv 0 \pmod p$;

$$\therefore\quad -n^{p-2}\Sigma s_1+n^{p-3}\Sigma s_1s_2-n^{p-4}\Sigma s_1s_2s_3+\ldots\equiv 0 \pmod p.$$

Since this is true for all integral values of n, it follows that each of the coefficients of the various powers of n must be a multiple of p.

$\therefore$ if p is prime, then the sum of the products r at a time

$$\{r=1, 2, 3, \ldots, (p-2)\}$$

of the numbers $1, 2, 3, \ldots, (p-1)$ is a multiple of p.

This is known as Lagrange's Theorem.

Example. Show that $12!\times 4!\equiv 2 \pmod{13}$.

By Wilson's Theorem, $12!+1=M(13)$.

Also $4!=26-2=M(13)-2$.

$$\therefore\quad 12!\times 4!=\{M(13)-1\}\{M(13)-2\}$$
$$=M(13)+2.$$

Q.E.D.

Examples 132

1. Consider the primes 17 and 19 and arrange (i) the numbers 2 to 15; (ii) 2 to 17 in pairs so that the product of each pair is (i) $M(17)+1$; (ii) $M(19)+1$.
2. Show that if p numbers are in A.P. and p is a prime, then no two of the numbers are congruent (mod p) provided the common difference of the A.P. $<p$.
3. (i) Verify that the sum of the products two at a time of the numbers less than 5 is 35 and so is divisible by 5.
   (ii) Verify that the sum of the products three at a time of the same numbers is also divisible by 5.
4. Writing $10!$ as $90 \,.\, 8!$ deduce that $2 \,.\, 8!+1=M(11)$.
5. From $12!+1=M(13)$ deduce that $11 \,.\, 8!+1=M(13)$.

6. Prove that $2 \cdot 28! + 1 = M(31)$.

7. Prove that $28! + 233 = M(29)$.

8. Deduce from Nos. 6, 7 that $28! + 233 = M(899)$.

9. Prove that $5!\,25! \equiv 1 \pmod{31}$.

10. Prove that if p is prime $(a+b+c+\ldots)^p = a^p + b^p + c^p + \ldots + M(p)$ and deduce Fermat's Theorem by supposing that $a = b = c = \ldots = 1$ and that there are n letters.

[Hint. It has been proved that if p is prime

$$(a+b)^p = a^p + b^p + M(p).$$

Hence

$$\begin{aligned}(a+b+c+\ldots)^p &= a^p + (b+c+\ldots)^p + M(p)\\ &= a^p + b^p + (c+\ldots)^p + M(p),\end{aligned}$$

and so on.]

First Degree Congruences

The congruence $3x \equiv 5 \pmod 7$ is called a *first degree congruence*, and the values of x which satisfy it are called roots.

Suppose first that the modulus is prime as in this congruence. The numbers which leave a remainder 5 on division by 7 are the successive terms of the A.P. whose first term is 5 and common difference 7, viz. the numbers 5, 12, 19, 26, 33, 40, 47, 54, ... To satisfy $3x \equiv 5 \pmod 7$, since x must be a whole number, we must select the terms of the A.P. which are multiples of 3.

Thus $3x$ can be 12, 33, 54, ..., giving $x = 4, 11, 18, \ldots$.

It is sufficient to give one root, since the others differ from it by multiples of 7, and *the root* of the congruence is that which <7.

Thus if $3x \equiv 5 \pmod 7$ the root is $x = 4$.

The above process, though always possible, would be extremely long and tedious for a congruence such as $75x = 68 \pmod{89}$.

In this case, since 75 is prime to 89 we can find numbers a and b such that $75a - 89b = 1$ by the process of finding H.C.F.

$$\begin{array}{l}
75\,)\;89\;(\,1\\
\quad\;\; \underline{75}\\
\quad\;\; 14\,)\;75\;(\,5\\
\qquad\qquad \underline{70}\\
\qquad\qquad 5\,)\;14\;(\,2\\
\qquad\qquad\quad\;\; \underline{10}\\
\qquad\qquad\quad\;\; 4\,)\;5\;(\,1\\
\qquad\qquad\qquad\quad \underline{4}\\
\qquad\qquad\qquad\quad 1
\end{array}$$

$$\begin{aligned}
14 &= 89 - 75\\
5 &= 75 - 5(89 - 75)\\
&= 6 \cdot 75 - 5 \cdot 89\\
4 &= 89 - 75 - 2(6 \cdot 75 - 5 \cdot 89)\\
&= 11 \cdot 89 - 13 \cdot 75\\
1 &= 5 - 4 = 19 \cdot 75 - 16 \cdot 89. \ldots \text{(i)}
\end{aligned}$$

If $$75x \equiv 68 \pmod{89},$$
then $$19 \cdot 75x \equiv 19 \times 68 \pmod{89}.$$
Also $$16 \cdot 89x \equiv 0 \pmod{89}.$$
$$\therefore\ x \equiv 19 \times 68 \equiv 1292 \equiv 46 \pmod{89}.$$

Alternatively (i) gives $19 \cdot 75 \cdot 68 = 68 + 16 \cdot 89 \cdot 68,$
i.e. $$19 \cdot 75 \cdot 68 \equiv 68 \pmod{89}.$$
But $$75x \equiv 68 \pmod{89}.$$
$$\therefore\ x \equiv 19 \times 68 = 1292 \equiv 46 \pmod{89}.$$

The root is $x = 46$.

If $ax = b \pmod{c}$ and a is not prime to c, but a and c have f as H.C.F., then b must have f as a factor also, and the congruence can be changed to $(a/f)x \equiv (b/f) \pmod{c/f}$. In such a case, a first degree congruence may have more than one root; e.g. $9x \equiv 6 \pmod{21}$ has roots 10, 17.

If b has not f as a factor, $ax \equiv b \pmod{c}$ is impossible.

But it has been shown that if a is prime to c the values 1, 2, ... $(c-1)$ for x all give different remainders on division by c, the numbers 1 to $(c-1)$ in some order or other; so that the remainder will be b for only one value of x less than c.

Examples 133

Solve the congruences Nos. 1, 2, 3.

1. $3x \equiv 2 \pmod{7}$. 2. $5x \equiv 7 \pmod{11}$. 3. $7x \equiv 6 \pmod{13}$.

4. Consider the congruences:

(i) $8x \equiv 6 \pmod{7}$; (ii) $8x \equiv 6 \pmod{14}$; (iii) $8x \equiv 6 \pmod{28}$.

Show that the first has one solution, but that the second has two solutions and the third none.

5. Solve the congruence $17x \equiv 1 \pmod{41}$.

6. Find the smallest positive solution of the congruence
$$7x \equiv 2 \pmod{23};$$
find also the smallest positive solution which also satisfies
$$x \equiv 2 \pmod{11}.$$ (N.U.J.B.)

7. Find x such that $x \equiv 3 \pmod{5}$ and $x \equiv 2 \pmod{11}$.
[Hint. $11x \equiv 33 \pmod{55}$ and $5x \equiv 10 \pmod{55}$;
$$\therefore\ 6x \equiv 23 \pmod{55}.]$$

8. Find the least positive value of x for which $4x \equiv 5 \pmod{7}$ and $6x \equiv 7 \pmod{13}$.

Solve the congruences.

9. $51x \equiv 35 \pmod{61}$. 10. $101x \equiv 96 \pmod{211}$.

11. $131x \equiv 128 \pmod{389}$. 12. $72x = 88 \pmod{103}$.

Congruences of Higher Degree

$ax^2+bx+c=0$ is a quadratic equation if $a \neq 0$.

$ax^2+bx+c \equiv 0 \pmod{m}$ is a quadratic congruence if $a \not\equiv 0 \pmod{m}$.

If a were congruent to $0 \pmod{m}$, i.e. were a multiple of m, the term ax^2 would disappear from the congruence.

Suppose $x=p$ is a solution of

$$ax^2+bx+c \equiv 0 \pmod{m},$$

then

$$ap^2+bp+c \equiv 0 \pmod{m};$$

$$\therefore\ a(x^2-p^2)+b(x-p) \equiv 0 \pmod{m}$$

or

$$(x-p)(ax+ap+b) \equiv 0 \pmod{m}.$$

Thus in addition to $x=p$ there can only be one other solution viz. that given by $ax+ap+b \equiv 0 \pmod{m}$.

Thus a quadratic congruence cannot have more than two incongruent roots.

Similarly a congruence of degree n can only have n incongruent roots, this being shown by raising the degree of the congruence by unity at each step.

Example I. Solve $x^2+9x \equiv 5 \pmod{13}$.

To make the coefficient of x even, add $13x$ to the L.H.S.:

$$x^2+22x \equiv 5 \pmod{13}.$$

Completing the square:

$$x^2+22x+121 \equiv 5+121 \pmod{13} = 126 \pmod{13}.$$

As 126 is congruent to a square, we must find the square to which it is congruent. As $126-2 \cdot 13 = 100$ we get $(x+11)^2 \equiv 10^2 \pmod{13}$;

$\therefore\ x+11 \equiv \pm 10 \pmod{13}$,

$x \equiv -1$ or $-21 \pmod{13}$, i.e. $\equiv 12$ or 5. These are the roots.

Check. For $x=12$. $12^2+9 \cdot 12 = 144+108 = 252 = 19 \cdot 13 + 5$.

For $x=5$. $5^2+9 \cdot 5 = 25+45 = 70 = 5 \cdot 13 + 5$.

Example II. Solve $3x^2+7x \equiv 14 \pmod{17}$.

The 3 must be got rid of. Now $7 \times 17 + 7 = 119+7 = 126$ is $M(3)$, and $14-17$ is $M(3)$.

The congruence is equivalent to

$$3x^2+126x \equiv -3 \quad \text{or to} \quad x^2+42x \equiv -1,$$

or to

$$x^2+42x+21^2 \equiv 441-1 \equiv 440 \pmod{17} \equiv 15 \pmod{17}.$$

$15+17k$ must be a square: take $k=2$ and $15+17k=49$.

$$\therefore\ (x+21)^2 \equiv 49 \pmod{17}.$$

This gives $x+21\equiv 7$ or -7,

$$x\equiv -14 \text{ or } -28,$$

i.e. $x\equiv 3$ or 6.

Check. If $x=3$, L.H.S. $= 27+21 = 48 = 2\times 17+14$.

If $x=6$, L.H.S. $=108+42=150=8\times 17+14$.

Examples 134. *Solve the congruences :*

1. $x^2+6x\equiv 6 \pmod 7$.
2. $x^2+3x\equiv 7 \pmod{11}$.
3. $x^2+8x\equiv 3 \pmod{17}$.
4. $2x^2+3x\equiv 2 \pmod 5$.
5. $3x^2+4x\equiv 7 \pmod{11}$.
6. $3x^2+5x\equiv 3 \pmod{13}$.
7. $4x^2+3x\equiv 3 \pmod 7$.
8. $x^2+2x\equiv 28 \pmod{71}$.

Miscellaneous Examples 135

1. Show that :

(i) $253\equiv 165 \pmod{11}$; (ii) $391\equiv 323 \pmod{17}$;
(iii) $670\equiv 437 \pmod{23}$; (iv) $3092\equiv 41 \pmod{113}$;
(v) $4658\equiv 5 \pmod{517}$; (vi) $48700\equiv 75 \pmod{389}$.

2. Verify the results :

(i) $48\equiv 18 \pmod 5$ and $78\equiv 63 \pmod 5$;
(ii) $(15 . 48)+(8 . 78)\equiv(15 . 18)+(8 . 63) \pmod 5$;
(iii) $p . 48+q . 78\equiv p . 18+q . 63 \pmod 5$;
(iv) $48 . 78\equiv 18 . 63 \pmod 5$.

3. Prove that if the common difference is prime to m, then m consecutive terms of an A.P. when divided by m give remainders that are all different.

4. Prove that $5^{1000}\equiv 1 \pmod{13}$.
[Hint. $5^2=2\times 13-1$.]

5. Prove that $3^{100}\equiv 4 \pmod 7$.
[Hint. Use $3^{100}\equiv 9^{50}\equiv 2^{50} \pmod 7$.]

6. Prove the congruences :

(i) $5^{6n}+2^{3n+1}\equiv 3 \pmod 7$. (ii) $3^{4n+2}\equiv 9 \pmod{41}$;
(iii) $7 . 75^n+2^{6n+2}\equiv 0 \pmod{11}$.

[Solution (i).

$$\begin{aligned} 5^{6n}&=(7-2)^{6n}\equiv 2^{6n} \pmod 7 = 8^{2n} \pmod 7\\ &=(7+1)^{2n}\equiv 1 \pmod 7,\\ 2^{3n+1}=2 . 2^{3n} &= 2(7+1)^n\equiv 2 \pmod 7;\\ \therefore\ 5^{6n}+2^{3n+1} &\equiv 3 \pmod 7.] \end{aligned}$$

7. Show that 99999 is a multiple of 41 and hence $10^5\equiv 1 \pmod{41}$.
Hence deduce $10^{10}\equiv 1 \pmod{41}$.
Also show $2^{10}\equiv -1 \pmod{41}$ and deduce $5^{10}\equiv -1 \pmod{41}$.
Further, since $3^{10}\equiv 9 \pmod{41}$ deduce $6^{10}\equiv -9 \pmod{41}$ and $30^{10}\equiv 9 \pmod{41}$.

8. If n is an integer greater than 1 show that n^2-n+1 cannot be the square of an integer. [Compare with $(n-1)^2$.]

9. If both n and $n+2$ are primes, prove that $(n+2)\{(n-1)!\}-2$ is divisible by $n(n+2)$. (N.)

10. (*a*) Find the least positive x for which both

$$4x \equiv 5 \pmod 7 \quad \text{and} \quad 6x \equiv 7 \pmod{13}.$$

(*b*) If x is a prime greater than 5, prove that $x^8 - 1$ is divisible by 480. (N.)

11. (*a*) If n is a positive integer, prove that $3^{4n+2} + 2^{6n+3}$ is divisible by 17.

(*b*) If n is a prime number, prove that

$$2(n+1) + 2^2(n+1)^2 + \dots + 2^{n-2}(n+1)^{n-2} + 2^{n-1}(n+1)^{n-1}$$

is divisible by n. (N.)

12. If p is a prime and r a positive integer, prove that

$$r!\,(p-r-1)! + (-1)^r \equiv 0 \pmod p.$$

If p is prime of the form $4q+3$, prove that p is a factor of either

$$(2q+1)!+1 \quad \text{or} \quad (2q+1)!-1. \qquad \text{(N.)}$$

13. If p and $p+2$ are both prime, prove that $2 \,.\, (p-1)! + 1 = M(p+2)$.

14. Show (by finding the numbers) that any value of x from one to ten makes x^2+x+41 a prime number. Find a value of x for which x^2+x+41 is not prime.

15. Show (by finding the numbers) that any value of x from one to ten makes $2x^2+29$ a prime number. Show that it is only for $x=4$ (or -3) that this formula gives the same number as x^2+x+41.

Find a value of x for which $2x^2+29$ is not prime.

16. If x, y, z are consecutive numbers, then $(x+y+z)^3 - 3(x^3+y^3+z^3)$ is $M(108)$.

17. If x and y are both prime to 5, prove that x^4+y^4 cannot be a perfect square. [Use Fermat's Theorem and consider possible forms of square numbers.]

18. Prove that x^{4n+1} ends with the same digit as x.

[Prove $x(x^{4n}-1) = M(10)$.]

19. Express 10^3 and 100^3 each as the difference of two squares.

Test Papers D

D.I

1. (i) Solve the equation $4^x = 10 \,.\, 2^{x-1} + 6 = 0$.

(ii) Prove that

$$(1+x+x^2+\dots+x^{2n})(1-x+x^2-x^3+\dots+x^{2n}) = 1+x^2+x^4+\dots+x^{4n}.$$

Resolve $1+x^2+x^4+x^6+x^8$ into two factors.

(iii) How many terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\dots$ must be taken so that their sum is within 1 per cent. of the sum to infinity? (L.)

2. (i) If α and β are the roots of the equation $ax^2+bx+c=0$ show that the roots of the equation

$$acx^2-(b^2-2ac)x+ac=0 \text{ are } \frac{\alpha}{\beta} \text{ and } \frac{\beta}{\alpha}.$$

(ii) Show that there are two values of λ for which the equation $\frac{\lambda}{x}+\frac{2}{x-1}+\frac{1}{x+1}=0$ has equal roots, and that the product of the two corresponding roots is unity. (L.)

3. Without assuming the formula for the number of permutations of n different things r at a time, prove the formula for the number of combinations of n different things r at a time.

Show that the number of combinations $n-2$ at a time of n things of which 3 are alike and the rest different is $\frac{1}{2}(n^2-5n+8)$. (L.)

4. (i) Find the sum of the coefficients in the binomial expansion of $(x+1)^n$, n being a positive integer.

(ii) The expression

$$(x+1)^n+(x+1)^{n-1}(x+2)+(x+1)^{n-2}(x+2)^2+\ldots+(x+2)^n$$

is rearranged as a series in ascending powers of x. Prove that the coefficient of x^r is

$$\frac{(n+1)!}{r!\,(n+1-r)!}(2^{n+1-r}-1). \qquad \text{(L.)}$$

5. (i) State the binomial series for $(1+x)^n$ where n is not a positive integer and $|x|<1$; state also the ratio of the $(r+1)$th term of the series to the rth term. Show that if x is positive this ratio is negative for all sufficiently large values of r. (L.)

(ii) Write down and simplify the series for $(1-x)^{-2}$ and show that the sum of the first m terms of this series is

$$\frac{1-x^m}{(1-x)^2}-\frac{mx^m}{1-x}. \qquad \text{(L.)}$$

6. Find to two places of decimals the root between 1 and 2 of the equation $x^3-4x+1=0$.

What inference can be drawn from the change of sign in the constant term of the equation if x is replaced by $x'+1$?

D.II

1. Prove that if $b^2>4ac$ the equation $ax^2+bx+c=0$ has real roots.

Show that the expression

$$\frac{(x-2)(x-6)}{(x-1)(x-3)}$$

where x is real, is capable of all real values. (L.)

2. If a, b, c are in geometric progression and $a-b$, $b-c$, $c-a$ are in harmonic progression, show that

$$b^2 = 2a(a+b). \qquad \text{(L.)}$$

3. Write down the expansion of $\log_e(1+x)$ in ascending powers of x, stating the range of values of x for which it is valid.

Deduce the expansion of $\log_e y$, where $y>0$, in ascending powers of $\dfrac{y-1}{y+1}$.

Show that, if

$$S_1 = \tfrac{1}{2} - \tfrac{1}{2}(\tfrac{1}{2})^2 + \tfrac{1}{3}(\tfrac{1}{2})^3 - \tfrac{1}{4}(\tfrac{1}{2})^4 + \dots,$$
$$S_2 = \tfrac{1}{5} + \tfrac{1}{3}(\tfrac{1}{5})^3 + \tfrac{1}{5}(\tfrac{1}{5})^5 + \tfrac{1}{7}(\tfrac{1}{5})^7 + \dots,$$

then $S_1 = 2S_2$. (L.)

4. (i) By induction, or otherwise, prove that the sum of n terms of the series

$$\frac{2}{3} + \frac{2\cdot 4}{3\cdot 5} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} + \dots$$

is equal to

$$-2 + \frac{2\cdot 4 \dots (2n+2)}{1\cdot 3 \dots (2n+1)}.$$

(ii) Given that

$$a_n = 2^{n+2} + 3^{2n+1},$$

simplify the value of the expression $a_{n+1} - 2a_n$. Hence prove that, if n is a positive integer, a_n is divisible by 7.

5. (i) Prove that

$$(\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\beta\gamma+\gamma\alpha+\alpha\beta) = \alpha^3+\beta^3+\gamma^3 - 3\alpha\beta\gamma.$$

(ii) Prove that, if $\alpha+\beta+\gamma=0$, $\alpha^2+\beta^2+\gamma^2=Q$, and $\alpha^3+\beta^3+\gamma^3=-R$, then α, β, γ are the roots of the equation $x^3 - \frac{1}{2}Qx + \frac{1}{3}R = 0$.

Find, in terms of Q and R, the values of $\alpha^4+\beta^4+\gamma^4$ and of $\alpha^5+\beta^5+\gamma^5$.

6. Show that if $\Delta \equiv \begin{vmatrix} a & b & b \\ p & a & b \\ q & r & b \end{vmatrix}$, then the value of the determinant formed by adding x to all the elements of Δ is of the form $A+Bx$ when $A=\Delta$ and $bB = \Delta - (a-b)^3$.

D.III

1. Write down the expansion for e^x in ascending powers of x, giving the general term. For what values of x is the expansion valid?

The first three terms in the expansion of $(a+bx+cx^2)e^{-x}$ are

$$2 - 5x + 6x^2.$$

Find a, b, c and show that the coefficient of x^r in this expansion is

$$(-1)^r(2r^2+r+2)/r! \qquad \text{(L.)}$$

2. Obtain the expansion for $\log_e\left(\frac{1+x}{1-x}\right)$ in ascending powers of x, and deduce that, if y is positive,

$$\left(\frac{y-1}{y+1}\right)+\frac{1}{3}\left(\frac{y-1}{y+1}\right)^3+\frac{1}{5}\left(\frac{y-1}{y+1}\right)^5+\ldots=\tfrac{1}{2}\log_e y.$$

By use of this expansion obtain the value of

$$\tfrac{1}{2}\log_e 11-\log_e 3$$

to six places of decimals. (L.)

3. In the binomial expansion for $(1+x)^n$ prove that the coefficients of x^s and x^{n-s} are equal, s being less than n and both of them positive integers.

In the expansion of $(1+x)^{2n}$ show that the sum of the last n coefficients is $2^{2n-1}-\frac{(2n-1)!}{(n-1)!\,n!}$. (L.)

4. If α, β are the roots of the equation

$$ax^2+bx+c=0,$$

find in terms of a, b, c, the values of (i) $\alpha^2+\beta^2$, (ii) $\alpha^3+\beta^3$.

Prove also that if $\alpha^r+\beta^r$ is denoted by S_r and if r is a positive integer then

$$aS_{r+2}+bS_{r+1}+cS_r=0. \qquad \text{(L.)}$$

5. If $a>b>0$, prove that the arithmetic mean of a and b is greater than their geometric mean.

Also show $(4a+b)/4>$ the harmonic mean of a and b.

6. If b is small compared with a, show that

$$\left(\frac{a-b}{a+b}\right)^{\frac{1}{n}}=\frac{na-b}{na+b}$$

approximately.

Deduce that, if the difference between p and q is small in comparison with their sum,

$$\left(\frac{p}{q}\right)^{-\frac{1}{n}}=\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}$$

approximately.

Use this to evaluate to three places of decimals, the fourth root of 0·98. (L.)

D.IV

1. (a) If $x^4+4x^3+ax^2+bx+25$ is the square of x^2+px+q, find the possible values of a, b, p, q.

(b) Eliminate x and y from the equations

$$x+y=a,\ x^2+y^2=b^2,\ x^3+y^3=c^3. \qquad \text{(L.)}$$

2. Show that the reciprocals of the roots of the equation

$$ax^4+bx^3+cx^2+dx+e=0$$

are given by the equation $ex^4+dx^3+cx^2+bx+a=0$.

Solve the equation

$$15x^4-16x^3-56x^2+64x-16=0$$

given that its roots are in harmonic progression. (L.)

3. Prove that $(ax^2+bx+c)/(a'x^2+b'x+c')$ (where all the symbols denote real numbers) can take all values if

$$b'^2>4a'c' \quad \text{and} \quad (a'c-ac')^2<(bc'-b'c)(ab'-a'b). \qquad \text{(L.)}$$

4. Prove that in general there are two distinct values of λ which allow $ax^2+2bx+c+\lambda(a'x^2+2b'x+c')$ to be expressed in the form $\mu(x+\nu)^2$.

When only one value is possible, prove that

$$(ac'-a'c)^2=4(a'b-ab')(b'c-bc'). \qquad \text{(L.)}$$

5. (i) Prove without expanding the determinants that

$$\begin{vmatrix} x & p & a \\ y & q & b \\ z & r & c \end{vmatrix} = \begin{vmatrix} p & r & q \\ x & z & y \\ a & c & b \end{vmatrix};$$

(ii) Solve completely the equation

$$\begin{vmatrix} x+2 & 3 & x \\ 1 & 4 & x \\ 7 & x & 1 \end{vmatrix} = 0. \qquad \text{(L.)}$$

6. Sum the infinite series :

$$\text{(i) } \frac{5}{6}+\frac{5\cdot 7}{6\cdot 9}+\frac{5\cdot 7\cdot 9}{6\cdot 9\cdot 12}+\dots;$$

$$\text{(ii) } 1+\frac{x}{2!}+\frac{2x^2}{3!}+\frac{3x^3}{4!}+\dots. \qquad \text{(L.)}$$

D.V

1. If $a_0x^3+3a_1x^2+3a_2x+a_3=0$ is such that

$$a_0a_2=a_1^2 \text{ and } a_1a_3=a_2^2,$$

prove that either the equation has three equal roots or it has roots proportional to $1, \omega, \omega^2$, the cube roots of unity.

2. (i) Find the modulus of $\dfrac{(2+3i)(3-4i)}{(6-4i)(15-8i)}$;

(ii) Show that the modulus of $ae^{i\theta}+be^{i\phi}$, when a, b, θ, ϕ are real is

$$\sqrt{\{a^2+b^2+2ab\cos(\theta-\phi)\}}.$$

3. Show from a graph that the equation $x^3-3x^2+x-4=0$ has only one real root and that this root is near to $x=3$.

By substituting $x=3+\epsilon$ and by successive approximation to the value of ϵ (or otherwise) find this root to 3 decimal places. (B.)

4. If a and b are positive, expand $\log_e a/b$

(i) in ascending powers of $(a-b)/b$,

(ii) in ascending powers of $(a-b)/a$,

obtaining the conditions under which the two series are valid.

Deduce, or prove otherwise, that, if powers of $a-b$ above the fourth can be neglected in comparison with a or b,

$$\log_e \frac{a}{b} = \frac{a^2-b^2}{2ab}\left[1-\frac{(a-b)^2}{6ab}\right].$$

5. (i) Prove that $(1-5x+4x^2)^{\frac{1}{3}} = 1-\frac{5}{3}x-\frac{13}{9}x^2+\ldots$ and find the coefficient of x^4.

[Assume $|x|<\frac{1}{4}$.]

(ii) Use the binomial theorem to calculate the fifth root of 1·002 correct to 8 decimal places. (N.)

6. Express

$$\begin{vmatrix} (a-a_1)^{-2} & (a-a_1)^{-1} & a_1^{-1} \\ (a-a_2)^{-2} & (a-a_2)^{-1} & a_2^{-1} \\ (a-a_3)^{-2} & (a-a_3)^{-1} & a_3^{-1} \end{vmatrix}$$

as a fraction in which both numerator and denominator are resolved into factors.

D.VI

1. (*a*) Which is greater, $2^{3^{41}}$ or $4^{3^{21}}$? Give reasons for your answer.

(*b*) If a, b, c, are unequal prove that $a^2+b^2+c^2$ can never be equal to $bc+ca+ab$. (L.)

2. What is meant by the statement "The series Σu_n is convergent"?

Σu_n is a series of positive terms. Prove that the series is convergent if (after some particular term) $\frac{u_{n-1}}{u_n}<\kappa$ where κ is a fixed number less than 1.

State and prove a corresponding divergency test.

Test for convergency the series whose nth terms are

(i) $n(n+1)(n+2)/2^n$; (ii) $1/(an+b)$. (L.)

3. Determine b and c so that the coefficients of x^3 and x^4 in the expansion of $(1+bx+cx^2)\log_e(1+x)$ may vanish.

Prove that with these values of b and c the error in taking

$$\frac{x+\frac{1}{2}x^2}{1+bx+cx^2}$$

for $\log_e(1+x)$ is $\frac{x^5}{180}$, neglecting powers of x higher than the fifth.

4. If α, β, γ are the roots of $x^3 - 21x + 35 = 0$, write down the values of $\alpha+\beta+\gamma$ and $\alpha\beta+\alpha\gamma+\beta\gamma$ and deduce that the product

$$(\alpha^2 + 2\alpha - 14 - \beta)(\alpha^2 + 2\alpha - 14 - \gamma)$$

has $\alpha^3 - 21\alpha + 35$ as a factor and so is zero.

Hence, given α is a root of the equation, what are the other two roots in terms of α?

5. Expand

$$\frac{x}{1 - e^{-x}} \equiv \left\{1 - \left(\frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \dots\right)\right\}^{-1}$$

as far as x^6, showing that the terms in x^3 and x^5 disappear.

6. Show that $\dfrac{x}{1 - e^{-x}} - \dfrac{x}{2}$ is a function whose value does not alter if the sign of x is changed.

[This shows that in the expansion of the previous example all the odd powers of x disappear except for $\frac{1}{2}x$.

The expansion is usually written

$$1 + \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_2 x^4}{4!} + \frac{B_3 x^6}{6!} - \dots$$

where B_1, B_2, B_3 are the first three of a set of numbers, called Bernoulli's numbers, which occur in the expansion of various other functions.]

From the result of Example 5 give the values of B_1, B_2, B_3.

D.VII

1. The first two terms of a geometric progression are a and b, b being less than a. If the sum of the first n terms is equal to the sum to infinity of the remaining terms, prove that $a^n = 2b^n$.

In this case prove that if a new series is formed by removing the first n terms of the original series, the sum of the first n terms of this new series is again equal to the sum of the remaining terms. (L.)

2. If a is positive show that $ax^2 + bx + c$ is positive for all values of x unless $at^2 + bt + c = 0$ has real roots and x lies between them.

Show that the expression $\dfrac{x^2 - 16x + 63}{x^2 - 18x + 80}$ can take all real values and find for what values of x the expression is negative. (L.)

3. (i) Write down the exponential series for a^x.

(ii) By using the logarithmic series show that when n is large

$$\left(1 + \frac{x}{n}\right)^n \simeq e^x. \quad \text{(L.)}$$

4. Show that if $-1 \leq x < 1$, where x is a real variable, the series

$$\sum_{n=1}^{\infty} \frac{(4n - 1)x^{n-1}}{n(n + 1)}$$

is convergent.

Assuming that x has a value in this range, sum the series. (L.)

5. Factorise the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

If $f(x)$ is a polynomial of degree less than three, prove that

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \begin{vmatrix} 1 & a & \dfrac{f(a)}{x-a} \\ 1 & b & \dfrac{f(b)}{x-b} \\ 1 & c & \dfrac{f(c)}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}. \qquad \text{(L.)}$$

6. Define the functions $\sinh x$, $\cosh x$ and show that

$$\sin ix = i \sinh x, \quad \cos ix = \cosh x.$$

If $x + iy = a \tan (u + iv)$, show that

$$\frac{x}{\sin 2u} = \frac{y}{\sinh 2v} = \frac{a}{\cos 2u + \cosh 2v}.$$

If x, y are the rectangular cartesian co-ordinates of a point P in a plane, show that, for any given value of v, P lies on the circle

$$x^2 + y^2 - 2ay \coth 2v + a^2 = 0. \qquad \text{(L.)}$$

D.VIII

1. A vertical blackboard is divided into squares by m equally spaced horizontal lines and n equally spaced vertical lines, where $m<n$. Prove that the *total* number of squares so formed is

$$\tfrac{1}{6} m(m-1)(3n-m-1).$$

2. (i) If $r^2x = aR + bR^3$, where a, b, R are positive and $R^2>r^2$, prove that $x>2\sqrt{(ab)}$.

(ii) If a, b, c, d are all greater than 1, prove that

$$8(abcd+1)>(a+1)(b+1)(c+1)(d+1).$$

(iii) By considering the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}$ and another sequence, prove that, if $n>2$,

$$n(n+1)^{\frac{1}{n}} - n < 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} < n - (n-1)n^{-\frac{1}{n-1}}.$$

3. (i) $x^3 + 3px^2 + 3qx + r = 0$ has three real roots α_1, α_2, α_3; $x^2 + 2kx + l = 0$ has two real roots, β_1 and β_2. If

$$\alpha_1<\beta_1<\alpha_2<\beta_2<\alpha_3,$$

prove that $2pk>q+l$.

(ii) In $x^3+3px^2+3qx+r=0$ we can put $x=y+a$ and have as a result $y^3+3p'y^2+3q'y+r'=0$. Prove that, if α is a real root for x, we can find a unique real value for a such that $\alpha - a = -\frac{r'}{q'}$, unless $\alpha = -p'$.

4. If a, b, c are three consecutive integers, prove that

$$\log_e b = \tfrac{1}{2}\log_e a + \tfrac{1}{2}\log_e c + \frac{1}{2ac+1} + \frac{1}{3}\frac{1}{(2ac+1)^3} + \frac{1}{5}\frac{1}{(2ac+1)^5} + \dots .$$

Use this to express log 41 in terms of log 2, log 3, log 5 and log 7.

5. Assuming $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \Big[\sin^{-1} x\Big]_0^{\frac{1}{2}} = \frac{\pi}{6}$, show by expanding $(1-x^2)^{-\frac{1}{2}}$ by the Binomial Theorem and integrating term by term that

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3\cdot 2^3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5\cdot 2^5} + \dots .$$

Find the value this gives to π, using the first 3 terms.

6. If $a_0z^3+3a_1z^2+3a_2z+a_3 \equiv A(z-p)^3+B(z-q)^3$ by equating coefficients of the powers of z show that a_0p+a_1, a_1p+a_2 and a_2p+a_3 can be expressed in terms of B, p and q, and deduce that

$$(a_0p+a_1)(a_2p+a_3)=(a_1p+a_2)^2.$$

Show also that p may be replaced by q in this relation.

Write the quadratic equation determining p and q as a two-row determinant equated to zero.

D.IX

1. If a, b, c are real constants, find the conditions that the expression ax^2+bx+c may be positive for all real values of the variable x.

If p is a real constant and x a real variable, find the ranges of values which can be assumed by the expressions

$$\text{(i)}\ \frac{(x+p)^2}{x+1};\quad \text{(ii)}\ \frac{(x+p)^2}{x^2+1}.$$

2. Prove that the arithmetic mean of n positive quantities is greater than or equal to their geometric mean.

The equation

$$a_0x^n - a_1x^{n-1} + \dots + (-1)^{n-1}a_{n-1} + (-1)^n a_n = 0,$$

in which all the a's are positive, has all its roots positive.

Show that

$$a_1a_{n-1} \geqslant n^2 a_0 a_n. \qquad \text{(L.)}$$

3. Find the values of a, b, c so that the coefficient of x^n in the expansion of $(a+bx+cx^2)e^x$ in ascending powers of x shall be $(n+1)^2/n!$ (L.)

4. If $\frac{x}{y}+\frac{y}{x}=a$, $\frac{y}{z}+\frac{z}{y}=b$, $\frac{z}{x}+\frac{x}{z}=c$, and x, y, z are all real, prove

(i) a^2, b^2, c^2, are each not less than 4.

(ii) If two of a, b, c, are equal to -2 the other must be $+2$.

(iii) $(a \pm \sqrt{a^2-4})(b \pm \sqrt{b^2-4})(c \pm \sqrt{c^2-4})=8$ where one of the ambiguous signs is opposite to the other two. (L.)

5. Sum to infinity the series:

(i) $1+\frac{1}{8}+\frac{1 \cdot 3}{2!\,8^2}+\frac{1 \cdot 3 \cdot 5}{3!\,8^3}+\dots$,

(ii) $\frac{3}{2!}+\frac{8}{3!}+\frac{15}{4!}+\frac{24}{5!}+\dots$,

(iii) $\frac{1}{1 \cdot 2}-\frac{x}{2 \cdot 3}+\frac{x^2}{3 \cdot 4}-\frac{x^3}{4 \cdot 5}+\dots$,

x being chosen so that the series is convergent. (L.)

6. If $g(x)$, $f(x)$ are two polynomials in x, the degree n of $f(x)$ being greater than that of $g(x)$, and if $f(x)$ has n distinct factors $(x-a_1) \dots (x-a_n)$, find constants $A_1 \dots A_n$ so that

$$\frac{g(x)}{f(x)}=\frac{A_1}{x-a_1}+\dots+\frac{A_n}{x-a_n}.$$

How is the result modified if $a_1=a_2$?

If
$$\frac{1}{x-a_4}=\frac{B_1}{x-a_1}+\frac{B_2}{(x-a_1)(x-a_2)}+\frac{B_3}{(x-a_1)(x-a_2)(x-a_3)}+\frac{B_4}{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}$$
find B_1, B_2, B_3, B_4. (L.)

ANSWERS

CHAPTER I (pp. 1 to 19)

Examples 1:

1. 24. **2.** -6. **3.** $\frac{1}{2}$. **4.** $1\frac{2}{5}$.
5. $17\frac{2}{3}$. **6.** $-\frac{13}{77}$. **7.** (i) 1·3; (ii) ·3. **8.** (i) 7; (ii) -5.
9. (i) $a+2b$; (ii) $ab/(a+b)$. **10.** $(b+d)/(a+c)$.

Examples 2:

1. $\frac{1}{2}$, 4. **2.** 1, -1. **3.** -5, -2. **4.** $2\frac{12}{17}$, $-\frac{7}{51}$.
5. 5, 2. **6.** 43, 33. **7.** 12, 6. **8.** $\frac{1}{2}$, $\frac{1}{3}$.
9. 8, 1. **10.** $3\frac{1}{2}$, $-\frac{1}{2}$. **11.** ·1, ·4. **12.** 2·43, 1.
13. 18, 4. **14.** 10, 2. **15.** 2, 1·3. **16.** $a+b$, c.
17. $c-b$, a.

Examples 3:

1. 42. **2.** 5 : 2. **3.** 8. **4.** 660 yd. **5.** 14.
6. 4, $3\frac{1}{2}$. **7.** 100 lb. **8.** 15 mi. **9.** $2\frac{1}{2}$ mi., 9.35 A.M.

Examples 4:

1. (i) 7; (ii) -6; (iii) 2; (iv) 4.
2. (i) 17, 1; (ii) 1, -1; (iii) $\frac{183}{74}$, $\frac{6}{74}$; (iv) 2, 5.
3. (i) $3x^2-2y^2$; (ii) 0; (iii) x^2+y^2.
4. $x:y:z=b_1c_2-b_2c_1:c_1a_2-c_2a_1:a_1b_2-a_2b_1$.
5. $x^2:x:1$ in same ratios as $x:y:z$ in No. 4.
6. $-\frac{17}{19}$, $\frac{33}{19}$. **7.** $\frac{47}{31}$, $\frac{53}{31}$. **8.** $-\frac{90}{83}$, $\frac{77}{83}$.

Examples 5:

1. (i) $xy=ab$; (ii) $x^2/a^2+y^2/b^2=1$. **2.** (i) $\frac{13}{6}$; (ii) ± 14.
3. (i) $s=vt-\frac{1}{2}at^2$; (ii) $v^2-u^2=2as$. **5.** $13y^2-66y+1=0$.
6. $13x^2-35x=68$. **7.** $p^2+q^2=a^2$.
8. $(3bc-4ad)^2=4(a^2-b^2)(4bd-3ac)$.
9. $abc+2fgh-af^2-bg^2-ch^2$.
11. $145a^2+63b^2-192ab-16a+24b-80=0$.

Examples 6:

1. 3, 4, 1. **2.** 4, 5, 4. **3.** 8, -6, 4. **4.** $8\frac{1}{2}$, $5\frac{1}{2}$, $\frac{1}{2}$.
5. -1, 2, 5. **6.** $6\frac{1}{2}$, -5, $-2\frac{1}{2}$, 13. **7.** -6, 12, 2.
8. 2, $4\frac{1}{2}$, $3\frac{1}{3}$. **9.** 9, 21, -4. **10.** 7, 5, 8.

Examples 7:

1. (i) $2\frac{3}{4}$, $5\frac{1}{16}$; (ii) $\frac{24}{7}$, $-\frac{13}{7}$; (iii) 4, 4. **3.** 1st and 3rd.
4. (i) Circumference from diameter; (ii) cm. from inches.
5. Join (0, 63) to (100, 126); 38.

Examples 8:

1. (i) 2; $y-2x=1$; (ii) $\frac{3}{2}$; $y=\frac{3}{2}x+6$.
2. No. (ii) is $2x-3y+1=0$. **3.** $\frac{3}{4}, \frac{3}{2}$. **4.** $15c/19$.
5. (i) $1+3t$, $-1+4t$; (ii) $2+t$, $1-2t$; (iii) $1+4t$, $-2+7t$.
6. (i) $7x+y=2\frac{1}{2}$; (ii) $3x-2y=27$.
7. (i) $x(d-b)-y(c-a)=ad-bc$; (ii) $dx-by=ad-bc$.
8. $2x-3y+13=0$. **9.** $\tan^{-1}\frac{4}{3}$, 5.
11. $x=2/(10+a)$, $y=-(20+3a)/(10+a)$ if $a\neq-10$; $a=-8$.
12. $\lambda=-3$.

Examples 9:

1. From left to right $x=60, 105$; $y=42, 133, 175$.
2. (i) 11, 7, 42; (ii) 28, 24, 14, 18; (iii) $2c+d$; $f-e$.
5. 15 : 9 : 4. **7.** 6, 10, 14. **8.** $\frac{11}{16}, \frac{7}{25}$.

Examples 10:

1. (i) 72; (ii) $12a$; (iii) $bcd/(c-b)$. **2.** $u/2a$, u/a.
3. (i) 8, 3; (ii) 6, 6. **4.** (i) $(a^2+b^2+c^2)/(a+b+c)$.
5. 1, -2. **6.** (i) -1; (ii) $b(a-3c)/a(2c-1)$.
7. (i) 3, 5; (ii) $\frac{1}{2}, 2\frac{1}{2}$. **9.** (i) 15; (ii) 4. **10.** 17, 8.
11. (i) 13; (ii) 11. **12.** (i) (2, 9); (ii) $(-\frac{1}{2}, 3)$; (iii) $(\frac{6}{7}, \frac{16}{7})$.
13. £75, £135, £135. **14.** (i) 3; (ii) -20.
15. (i) -7; (ii) 64; (iii) $2a^2$. **16.** $3x+2y=17$; $-\frac{3}{2}$.
17. (80, 40), $(-80, -40)$. **18.** $\{a(2b-a)/3b, (a+b)/3\}$.
19. 10, 3, 16.

CHAPTER II (pp. 20 to 52)

Examples 11:

1. $(x-5)(x-6)$. **2.** $(x-12)(x+1)$. **3.** $(x-21)(x+10)$.
4. $(2x+1)(x+2)$. **5.** $(3x+7)(x-2)$. **6.** $(3x+5)(x-1)$.
7. $(12a+13b)(12a-13b)$. **8.** $(1+15x)(1-15x)$.
9. $(3+y^2)(1-2y^2)$. **10.** $(x+7)(2x-5)$. **11.** $(3y+2)(y-2)$.
12. $(2-x)(1+8x)$. **13.** $(5s-t)(s-5t)$. **14.** $(2-5y)(1-7y)$.
15. $(2-7x)(2+3x)$. **16.** $(a+b+c)(a+b-c)$.
17. $(5z-2a)(3z+a)$. **18.** $(9+8x)(8-9x)$.
19. (i) $(x+\sqrt{3})(x-\sqrt{3})$; (ii) $(2x+\sqrt{11})(2x-\sqrt{11})$;
(iii) $(x+y\sqrt{5})(x-y\sqrt{5})$.
20. (i) $(x-1+\sqrt{3})(x-1-\sqrt{3})$; (ii) $(2x-5+\sqrt{7})(2x-5-\sqrt{7})$.

Examples 12:

1. (i) 49; (ii) $\frac{169}{4}$; (iii) 25. **2.** (i) $\frac{25}{4}$; (ii) $121b^2/4$; (iii) $100z^2$.

Examples 13:

1. (i) $3\frac{1}{2}$, 4; (ii) $2\frac{1}{2}$, -7; (iii) 2, $\frac{4}{3}$; (iv) 7, 0.
2. (i) 1·72, $-$·39; (ii) 8·12, $-$·12; (iii) ·14, $-$·80; (iv) 1·74, $-$6·74.
3. (i) 1·65, $-$3·45; (ii) $\frac{1}{2}$, $-3\frac{1}{2}$. **4.** (i) Identity; (ii) 1, -2.
5. (i) 2·28, $-$4·19; (ii) 2·94, $-$10·22.
6. (i) ·35, $-$1·63; (ii) $\frac{4}{3}$, $-\frac{5}{2}$. **7.** (i) 1, $\frac{9}{2}$; (ii) 5·21, ·288.
8. (i) $-$4·3, $-$57·9; (ii) No roots. **9.** (i) 3, $\frac{1}{8}$; (ii) 9·33, ·67.
10. (i) 5, -1; (ii) 6·46, $-$·46.
11. (i) 4 : 3, 7 : 2; (ii) 2 : 1, -4 : 1. **12.** (i) 5, $-\frac{1}{2}$; (ii) $-\frac{1}{3}$, $\frac{7}{4}$.
13. (i) 8, -5; (ii) ·192, $-$5·192. **14.** (i) 11, -10; (ii) 5, -2.
15. (i) 24, -20; (ii) 2, $-\frac{7}{17}$. **16.** (i) 0, -1; (ii) ·618, $-$1·618.
17. 37, 38. **18.** 60, 62. **19.** 39, 41. **20.** 8, 10. **21.** 12, 15.
22. 42 ft.

Examples 14:

1. (i) 25; (ii) 165; (iii) $529b^2$; (iv) $36a^2b^2$; (i) and (iv). **2.** -4, -4.
3. (i) rational; (ii) irrational; (iii) none; (iv) equal.
4. (i) 14; (ii) 24. There is a factor 2.
5. (i) $x-4\pm\sqrt{7}$; (ii) $x+2\pm\sqrt{5}$.
7. (i) $x+\frac{1}{2}\pm\dfrac{\sqrt{5}}{2}$; (ii) none; (iii) none; (iv) $x-\frac{1}{2}\pm\dfrac{\sqrt{5}}{2}$.
9. (i) $-10\pm\sqrt{50}$; (ii) $(x+10)^2=-50$.
10. (i) -1, -56, 113, 225; (ii) 4, 2, 12, 8; (iii) $-\frac{7}{2}$, 2, $\frac{33}{4}$, $\frac{17}{4}$;
(iv) -1, -3, 7, 13.
11. (i) $(9b^2-8ac)/a^2$; $(9b^2-16ac)/a^2$; (ii) $81b^2/2a^2$, 0.
12. (i) $4p^2-2q$; (ii) $16p^4-16p^2q+3q^2$.
13. (i) $-p$; (ii) q; (iii) p^2-3q; (iv) $p^4-4p^2q+2q^2$.
16. p, $-q$, $p^4-4p^2q+2q^2$.

Examples 15:

1. (i) $6x^2+5x-50=0$; (ii) $x^2+12x+13=0$.
2. (i) $x^2-(3a+2b)x+6ab=0$; (ii) $x^2-2ax+a^2-p=0$.
3. $x^2-12x+4=0$. **4.** $qx^2+px+1=0$.
5. (i) $x^2-(p^2-2q)x+q^2=0$; (ii) $qx^2-(p^2-2q)x+q=0$.
6. $x^2-4\alpha x+3\alpha^2=0$. **7.** $acx^2+b(a+c)x+(a+c)^2=0$.
8. $x^2+(p-2)x+q-p+1=0$.
9. (i) 4, 64; (ii) 6, 216; (iii) $\frac{3}{5}$, $\frac{27}{125}$, $p=-510$.
10. (i) $x^2-2px+4q=0$; (ii) $4x^2-2px+q=0$;
(iii) $qx^2-(p^2-2q)x+q=0$; (iv) $x^2-(p+q)x+pq=0$.

Examples 16:

2. $2(\frac{5}{6})^2=2(\frac{5}{6}-\frac{3}{2})^2+\frac{1}{2}$. **4.** $-\frac{3}{2}$. **5.** $y=4x$.

Examples 17:

1. Vertex at (i) $(\frac{3}{2}, \frac{19}{4})$; (ii) $(\frac{3}{2}, -\frac{15}{2})$; (iii) $(-\frac{2}{5}, -11\frac{4}{5})$; (iv) $(-2, 10)$; (v) $(-\frac{3}{4}, 6\frac{1}{8})$; (vi) $(\frac{2}{5}, 15\frac{4}{5})$.
2. $-7<k<7$. 3. $-\frac{16}{3}$. 5. $2<x<5$. 6. $3<x<4$.

Examples 18:

1. $(x+2)^2+3$; min. 3 if $x=-2$.
2. $(x-4)^2-15$; min. -15 if $x=4$.
3. $-(x-\frac{3}{2})^2+\frac{9}{4}$; max. $\frac{9}{4}$ if $x=\frac{3}{2}$.
4. $(x-\frac{1}{2})^2+2\frac{3}{4}$; min. $2\frac{3}{4}$ if $x=\frac{1}{2}$.
5. $-4(x+\frac{1}{2})^2+9$; max. 9 if $x=-\frac{1}{2}$.
6. $2(x-3)^2+5$; min. 5 if $x=3$.
7. $3(x+1)^2-4$; min. -4 if $x=-1$.
8. $(x+h)^2+k-h^2$; min. $k-h^2$ if $x=-h$.
9. $a\left(x+\dfrac{b}{a}\right)^2+\dfrac{ac-b^2}{a}$; $a>0$, min. $\dfrac{ac-b^2}{a}$ if $x=-\dfrac{b}{a}$. 10. Each is 5.
11. $3<x<7$; max. 4 if $x=5$. 12. $k>-\frac{1}{3}$.

Examples 19:

2. 4. 4. 48. 5. $p(2hx+h^2)+qh$. 6. (i) even; (ii) odd.
7. $2(ax^2+c)$, $2bx$. 9. $6x^2+10xy+4y^2+10$; $14x+18y$.

Examples 20:

1. -4. 5. -2; $(3, 0)$ and $(0, 6)$. 6. $\frac{3}{4}$; -1 and $+1$.
7. 100 ft.; $2\frac{1}{2}$ sec.; upward velocity. 8. $\frac{5}{2}$. 10. 3 ft.

Examples 21:

1. $(a^2+b^2)(a-b)$. 2. $(px+qy)(rx-sy)$. 3. $(b+a)(x-a)$.
4. $(al-n)(l-m)$. 5. $(3x+y)(x+2)$.
6. $(a+b)(a+1)(a-1)$. 7. $(a-b)(a+b-2)$.
8. $(2x-y)(2x+y+2)$. 9. $(x+y)(x-y-z)$.
10. $(x+b-c)(x-b+c)$. 11. $(x+b+c)(x+b-c)$.
12. $(a+c-b)(a-c-3b)$. 13. $(x^2+a^2-b^2)(x^2-a^2+b^2)$.
14. $(3a+b)(a-3b)$. 15. $(x-y^2+4)(x-y^2-4)$.

Examples 22

1. (i) $(x+y)(x^2-xy+y^2)$; (ii) $(x-2y)(x^2+2xy+4y^2)$; (iii) $(x+3y)(x^2-3xy+9y^2)$.
2. (i) $(2a+1)(4a^2-2a+1)$; (ii) $(5ab+4c)(25a^2b^2-20abc+16c^2)$; (iii) $7x(y-z)(y^2+yz+z^2)$.
3. (i) $(x^2+a^2)(x^4-a^2x^2+a^4)$; (ii) $(x+a)(x-a)(x^4+a^2x^2+a^4)$; (iii) $4c^3(x-2y)(x^2+2xy+y^2)$.
4. (i) $2a(a^2+3b^2)$; (ii) $2b(3a^2+b^2)$.
5. $(7x+y)(x-5y)$. 6. $(3x^2-7)(x^2-8)$. 7. $(a+b)(c^2+d^2)$.
8. $(a-b)(c-d)(c^2+cd+d^2)$. 9. $(x+b+c)(x+b-c)$.

10. $(x+b-c)(x-b+c)$. 11. (i) $(2a+b)^3$; (ii) $(2x-3y)^3$.
12. (i) $(a^2+ab+b^2)(a^2-ab+b^2)$.
14. (i) $(a^2+b^2+ab\sqrt{3})(a^2+b^2-ab\sqrt{3})$; (ii) $(a^2+b^2+ab)(a^2+b^2-ab)$.

Examples 23:

1. $(3, 6), (-2, 1)$ 2. $(-23, -10), (4\frac{3}{5}, -\frac{4}{5})$.
3. $(5, 3), (1, \frac{1}{3})$. 4. $(14, 9), (-\frac{5}{2}, -2)$. 5. $(11, -11), (-3, 3)$.
6. $(4, 1), (3, -1)$. 7. $(20, 7), (-6, -1)$.
8. $(3, 14)(-\frac{331}{86}, -\frac{1447}{86})$. 11. 121, 40 yd. 12. 15, 8 yd.
13. 37 ft. 14. 6, 8, 10 yd.

Examples 24:

1. 5, 3. 2. 11, 9. 3. 7, 4 or $-7, -4$.
4. $-\frac{255}{32}, \frac{1}{4}\sqrt{799}$. 5. 42 ; 6, 7. 6. 2, 1 or $-2, -1$.

Examples 25:

1. $(2, 1), (-2, -1), \pm\left(\frac{4\sqrt{15}}{15}, -\frac{9\sqrt{15}}{15}\right)$.
2. $\pm(3, 4), \pm\left(\frac{6\sqrt{13}}{13}, \frac{17\sqrt{13}}{13}\right)$.
3. $\pm(2, 3), \pm\left(\frac{3\sqrt{74}}{37}, \frac{2\sqrt{74}}{37}\right)$.
4. $\pm(3\sqrt{2}, \frac{1}{2}\cdot 3\sqrt{2})$. 5. $\pm\left(\frac{\sqrt{28}}{3}, \frac{\sqrt{20}}{3}\right)$.
6. $\pm(4\frac{1}{2}, \frac{1}{2})$. 7. $\pm(2{\cdot}058, {\cdot}487)$. 8. $(5, 1), (-1, -5)$.
10. $(7x-6y)^2-y^2$; $(3x-7y)^2+(y\sqrt{2})^2$.

Examples 26:

1. (i) $\sqrt{18}$; (ii) $\sqrt{147}$; (iii) $\sqrt{60}$; (iv) $\sqrt{150}$.
2. (i) $3\sqrt{3}$; (ii) $9\sqrt{2}$; (iii) $2\sqrt{21}$; (iv) $12\sqrt{3}$.
3. (i) 3·58 ; (ii) 3·22 ; (iii) 5·10 ; (iv) 1·91 ; (v) ·30 ; (vi) ·78.
6. $\sqrt[12]{432}$. 8. $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$.

Examples 27:

1. (i) $-4, \frac{7}{3}$; (ii) $-a, 1$; (iii) $\frac{c}{a}, \frac{7c}{3a}$; (iv) $3p+q, 3p-q$.
2. (i) $\frac{1}{2}(5\pm\sqrt{53})$; (ii) $\frac{1}{4}(-7\pm\sqrt{33})$; (iii) $\frac{1}{5}(8\pm\sqrt{14})$; (iv) $\frac{13}{9}, \frac{13}{9}$;
(v) $\frac{1}{3a}(-6\pm\sqrt{37})$; (vi) $\frac{7b}{8a}, -\frac{2b}{17a}$.
3. (i) $x^2-4x+1=0$; (ii) $x^2+5x+3=0$.
4. (i) $(\frac{3}{2}, -2)$; (ii) $(\frac{3}{2}, -\frac{11}{6})$; (iii) $(\frac{3}{2}, -\frac{7}{4})$; $2x-3$ is factor.
5. $(2x+5)/(2x+3)$. 7. (i) $14\frac{3}{4}$; (ii) $3, -1\frac{1}{8}$. 8. 5 h.-c., 13s.
9. (i) $(x-4)(x-1)$; (ii) $(x^2+4)(x^2+1)$; (iii) $(x^2+x+2)(x^2-x+2)$.
10. 4·19, $-1{\cdot}19$. 11. (i) $(4, -1), (-1, 4)$; (ii) $(4, -1), (-\frac{64}{17}, \frac{49}{17})$.

12. £7 10s., £3 12s. 13. (i) $3x^2-10x-8=0$; (ii) (a) -4, $\frac{2}{3}$, (b) 2, $\frac{4}{3}$.
14. 60, 40 ch.
15. $12-(3-x)^2$; 3, 12; $-6+(3+x)^2$; -3, -6 least value.
16. 2·6 c.c.; 6·2. 17. 84. 18. (i) 5·06; (ii) 1·41 or $-3·91$.
19. 240 mi. 20. (i) $2x^2+5x-33=0$; $x^2-4x+1=0$; (ii) 2 or $1\frac{1}{2}$.
21. (i) $(x-a)(7y+4b)$; (ii) $(2a^2+3b^2-4ab)(2a^2+3b^2+4ab)$.
22. $(ab+bc-ac)/(a+c-b)$. 23. 40, 60. 24. $\frac{6}{7}$.
26. $(a/b+2+2b/a)^2$. 27. 8, 6 or 12, 5. 29. $\Sigma a^2+2\Sigma bc$.
30. Product $=4(a+b)c$.; $2kc$. 31. 89. 32. 45. 33. 5, $4\frac{8}{9}$.
34. 7, 8, 9, 10 or 1, 2, 3, 4. 35. 972 sq. in. (sides 36, 27 in.). 36. 4.
37. 2, 1, 6. 39. (ii) $x=-\frac{1}{4}$; $(-\frac{1}{4}, \frac{7}{8})$. 40. 3, -2, 5; $(\frac{1}{3}, 4\frac{2}{3})$.
41. $(3/2a, 9/4a)$. 42. (56, 22), $(26\frac{2}{3}, 0)$. 43. (8, 7), $(-7, -8)$.
44. (8, 10), $(-\frac{10}{9}, \frac{8}{9})$. 45. (5, 3), $(-5\frac{4}{5}, -4\frac{1}{5})$.
46. (5, 4), $\left(-\frac{8}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$. 47. $(\frac{5}{3}, 6)$, $(2i, -5i)$.
48. $(\pm 8, \pm 3)$. 49. (8, 3), $(-8, -3)$, $(8, -3)$, $(-8, 3)$.
50. (8, 3), (3, 8), $(8, -3)$, $(-3, 8)$.
51. (i) $x^2-16x+25=0$; (ii) $5x^2-2x-1=0$; (iii) $5x^2+16x+5=0$.
52. $\frac{4}{3}$. 53. $\pm a\sqrt{(1+m^2)}$. 56. $\sqrt{3}+\sqrt{5}$; $\sqrt{7}+1$.
57. (i) $\sqrt{7}+\sqrt{4}$; (ii) $3-\sqrt{6}$; (iii) $\sqrt{7}-\sqrt{3}$ or minus these.
61. 4. 62. 5. 63. 13, 5. 64. 10.
65. 800. 66. 8. 67. 5. 68. 481.
69. 8. 70. 8. 71. $\frac{1}{3}(4a-b)$, $\frac{1}{3}(4b-a)$. 72. 7; 7, 3.
74. $x+\frac{1}{x}=-2$ repeated; $x=-1$. 75. $\frac{1}{2}$, 2.
76. $x+\frac{1}{x}=2$; $x=1$.
77. $x-\frac{1}{x}=3$ or -1; $x=\frac{1}{2}(3\pm\sqrt{13})$ or $\frac{1}{2}(-1\pm\sqrt{5})$.
78. (6, 1), (1, 6), $(-\frac{6}{5}, -\frac{3}{5})$, $(-\frac{3}{5}, -\frac{6}{5})$.
79. $p=7$, $q=10$ giving (5, 2) or (2, 5);
$p=-4$, $q=-1$ giving $-2\pm\sqrt{5}$, $-2\mp\sqrt{5}$.
80. $p=6$, $q=1$ giving $3\pm\sqrt{8}$, $3\mp\sqrt{8}$;
$p=-1$, $q=-6$ giving $(2, -3)$, $(-3, 2)$.
81. If $bcx=t$ then $t^2=\frac{1}{3}abc$; abc must be positive.
82. $x=y=\pm\sqrt{(p-\frac{1}{2}q)}$, $z=\mp\frac{1}{2}q/\sqrt{(p-\frac{1}{2}q)}$.
83. k^2+k+1 must be $+$.
84. $a=3$ or $-\frac{1}{2}$; in 2nd case, square/2.
85. $\pm k=a\sqrt{3}$. 86. $k>1$.
87. $acx^2+b(a+2c)x+(a+2c)^2=0$.
90. $x=a$ or $(4-3a)/(3-2a)$. 91. $x^2-(p+6)x+3p+q+9=0$.
92. $a^3+b^3=3abc$. 94. $2x^2-6x+9=0$.

CHAPTER III (pp. 53 to 81)

Examples 28

1. (i) $V = aD^3$; (ii) $A = bMM'/D^2$; (iii) $L = \dfrac{cA}{D^2}$; (iv) $R = a + bV^2$;
(v) $R = al/d^2$; (vi) $S = br(r + h)$.
2. (i) $a = \cdot 524$; (iv) $a = 1200$, $b = \frac{1}{4}$; (vi) $b = 6\cdot 28$.
3. $y = \frac{25}{6}x^2$; 54. **4.** $x > 4$.
5. (i) multiply by $\sqrt{2}$; (ii) divide by $\sqrt{2}$.
6. (i) $V \propto h$; (ii) $V \propto r^2$; (iii) $t \propto \dfrac{1}{s}$. **7.** Change 42 to 45.
8. $y \propto x^2$; $x \propto \sqrt{y}$; $x = 2\sqrt{5}$. **9.** (i) 25 m.; (ii) $\sqrt{2}/10$ mm.
10. $w = kd^2$; (*a*) (i) 27 oz., (ii) 36·75 oz.; (*b*) (i) 5 in., (ii) 6·3 in.
11. (*a*) (i) 6·37 kg.; (ii) ·71 kg.; (*b*) 2·83 mm.
12. 12. ·4; 12 gm. **13.** $W^2 \propto Z^3$. **14.** $200bc/(b + c)^2$.
16. (i) 54 gm.; (ii) 3 cm.; (iii) 1·6 cm.

Examples 29:

1. (ii) v^2 against r, or log v against log r;
(iii) n^2 against $\dfrac{1}{r}$, or log n against log r;
(iv) S against $1/(3 + x)$, or log S against log $(3 + x)$;
(v) $(5 - y)$ against x^2, or log $(5 - y)$ against log x;
(vi) y^2 against x^3, or log y against log x.

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2. (i) $y = 2x^2$; (ii) $xy = 100$; (iii) $y = 10^x$;
(iv) $y^3 = x^2$; (v) $y = 10(\frac{1}{2})^x$; (vi) $y^2 = 10{,}000/x^3$.
4. $y = 64(\frac{1}{2})^x$.
5. $s = 1\cdot 8t^2$. **8.** $T \propto l^2$. **11.** $D = \cdot 2 + \cdot 053W$.
12. $R = 9\cdot 5 + \cdot 03T$. **13.** $k = 13$, $n = \frac{1}{2}$.
14. $a = 130$, $b = 335$, $c = 58$.

Examples 30:

3. $x = \frac{1}{3}(p + q + r)$. **4.** No (i). **5.** $r = (b^2 - 3ac)/3a$.

Examples 31

1. $y = x - 1$, $x = 2$; $1 \mp 2\sqrt{2}$.
2. $x = 1$, $x + 2y = 3$; max. $-(\sqrt{3} - 1)$, min. $(\sqrt{3} + 1)$.
3. $x + 2 = 0$, $2y - x = 3$; max. $-\sqrt{2} + \frac{1}{2}$, min. $\sqrt{2} + \frac{1}{2}$.
4. $x + 1 = 0$, $x + y - 2 = 0$; no max. or min.
5. $x - 1 = 0$, $y = x + 1$; max. 0, min. 4.
6. $x = 0$, $x + y + 1 = 0$; no max. or min.

Examples 32:

8. $\frac{1}{3} \leq y \leq 3$. **10.** $a > 0$; $b^2 \leq 4ac$. **11.** 3·13.
13. If k is negative, y_2 is max. If k is positive, y_1 is max.

Examples 33:

6. $q=0$.

Examples 34:

4. 2·81. 5. ·69; $-1\cdot17$. 6. 2·73.
7. 3·115. 8. 1·17. 9. ·1603 radians (9° 11′).

CHAPTER IV (pp. 82 to 96)

Examples 35:

1. (i) 2; (ii) 5; (iii) 2; (iv) 3; (v) $\frac{1}{6}$; (vi) $\frac{1}{81}$; (vii) 1; (viii) $\frac{1}{8}$; (ix) $\frac{1}{2}$; (x) 2; (xi) $\frac{1}{3}$; (xii) $\frac{1}{5}$.
2. (i) 4; (ii) $\frac{1}{9}$; (iii) $\frac{1}{32}$; (iv) $\frac{1}{8}$; (v) 1000; (vi) 32; (vii) 32; (viii) 64; (ix) 64; (x) 32; (xi) $\frac{1}{125}$; (xii) $\frac{1}{10}$.
3. (i) a^3x^5; (ii) $\dfrac{1}{a^{14}}$; (iii) $\dfrac{1}{x^{12}}$. (iv) $8a^9$. (v) $\dfrac{1}{4x^6}$. (vi) $27x^{12}$; (vii) $\dfrac{4}{x^6}$; (viii) $\dfrac{x^{12}}{27}$; (ix) $\dfrac{1}{x^3y^4}$; (x) $\dfrac{6a^4}{b}$; (xi) $\dfrac{4y^2}{x^2}$; (xii) $-\dfrac{2x^2}{y^4}$.

Examples 36:

1. (i) $10^{\cdot3010}$; (ii) $10^{\cdot4771}$; (iii) $10^{\cdot3495}$; (iv) $10^{\cdot9956}$; (v) $10^{\cdot2641}$; (vi) $10^{1\cdot1673}$.
2. (i) $5=10^{\cdot6990}$; (ii) $6\cdot31=10^{\cdot8}$; (iii) $20=10^{1\cdot3010}$.
3. (i) $1\cdot6=\log 39\cdot81$; (ii) $2\cdot902=\log 798$; (iii) $\log 4=2\log 2$.

Examples 37:

1. 2^{-7}; $2^{\frac{3}{2}}$. 2. $20=\text{antilog } 1\cdot3010=10^{1\cdot3010}$.
3. (i) $3=\log_5 125$ or $\log 125=3\log 5$ (any base); (ii) $\log 81=\frac{4}{3}\log 27$; (iii) If $\log_a k=4$, then $\log_a k^3=12$.
4. (i) $169=13^2$; (ii) $64=16^{\frac{3}{2}}$; (iii) $a^{12}=(a^2)^6$. 5. 4·8.
6. (i) ·3939; (ii) 300. 7. $2x=3y$. 8. (i) $x=\sqrt{3}$; (ii) $x=125$.
9. $\cdot75=10^{-\cdot1249}$. 10. (i) $\bar{1}\cdot8$; (ii) $-\cdot4192$.

Examples 38:

1. If $x=\log_a p$, $y=\log_a q$, $z=\log_a r$, then $x+y-z=\log_a(pq\div r)$.
2. (i) $y=2x$; (ii) $y=6x$. 3. 1·465, ·6825.
4. (i) Say $800\div32=5^2$; (ii) $10^3=75\times40\div3$. If R.H.S. $=x^3$, then $x=10$.
5. 100. 6. 12. 7. 0. 9. (i) $a^2=bc/d$; (ii) $a^3=d/b^2$.
10. (ii) $a^{2p}=(2a)^q$.

Examples 39:

1. 9×10^{-14} gm. 3. $2\cdot3\times10^{15}$ miles. 4. 7·020.
5. $-\cdot1988$. 6. $-\cdot769$. 7. $x\simeq-1$ or 2 (1·995).
8. $x\simeq\pm\cdot826$ or $\pm\cdot656$. 9. $x\simeq1\cdot35$, $y\simeq1\cdot71$.
10. $x=7\cdot08$, $y=3\cdot98$.

Examples 40:

1. $\dfrac{x^6}{4}$. 2. $\dfrac{1}{9x^4}$. 3. $27x^6$. 4. $3/y^{2\frac{1}{2}}$.
5. $y^{\frac{5}{2}}z^{\frac{3}{2}}$. 6. (i) 4^x; (ii) $1/3^{2a}$.
7. (i) $a^{4x}-a^{x-3y}$; (ii) $a+b+\surd(ab)$.
8. (i) $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$; (ii) $9x-4y^{\frac{2}{3}}+16y^{\frac{1}{3}}z^{\frac{1}{4}}-16z^{\frac{1}{2}}$.
9. -5. 10. $2\frac{1}{16}$. 11. 7. 13. (i) 3; (ii) 4; (iii) $\frac{3}{2}$.
13. (iv) -3; (v) -3; (vi) -3. 14. ·8260. 15. 4·292.
16. 19·94. 17. 0·836. 18. 1·107. 19. 2·838.
20. 0·01047. 21. 3·817. 22. $(3x+7)^2$; 1 or $16\frac{5}{9}$.

Examples 41:

5. 1·585. 7. $(2, -1)$. 8. $x \simeq 1·55$. 10. $x \simeq 1·46$.

Examples 42:

1. $\frac{1}{2}$, 2, 8; 5·66. 2. (4, 2).
4. $a=\frac{10}{3}$. O between Q and R with $OR \simeq ·6$; ·18, $6\frac{2}{3}$.
5. 0, 1, $\simeq 4·25$; $0<x<1$ and $x>4·25$.

Examples 43:

1. (iv) 1·099; (v) 1·609.
2. (i) at (0, 1); at right angles. (ii) ·693, 1·386; 1·099, 3·297.
4. Equal and opposite ($\pm$·69). 6. ·62. 7. ·24.

Examples 44:

1. 6. 2. (i) 2; (ii) 5. 3. $x=3y$. 4. (i) 9, 4; (ii) 9, 4, a.
5. (i) 2·578; (ii) ·5544. 6. (i) Ordinates to line $y=4$; (ii) $\frac{3}{7}$.
7. 5·623; 5·624 by log's, 5·623 by antilog's.
8. (iii) $5^4=625$; (iv) $8^{\frac{5}{3}}=32$; (v) $(a^5)^5=a^{25}$. 9. 10^{-4} secs.
10. 1·2, 3. 11. (i) $x^2y^5=17·06$; (ii) $-\frac{1}{2}$, ·131.
13. ·599. 14. 2·2, 4·1. 15. (i) $n=10$; (ii) $n=100$. 16. 2.
17. (i) $3\surd 30$; (ii) 2; (iii) ·06577. 18. 5·065. 19. $x \simeq 1·58$.
20. $\frac{1}{6}(z+w)$. 21. (b) $\simeq 1229$. 22. 35,000; 190,100.
23. 167. 24. 26, 15. 25. 35,007,000; 1905·2. 26. 126.
27. (i) 2^a, $2^{-\frac{1}{2}a}$, 2^{-a}; 2^{-a}, $2^{-\frac{1}{2}a}$, 2^a. (ii) 167. 29. $2a/x$, $4(a^2/x^2-1)$.

CHAPTER V (pp. 97 to 117)

Examples 45:

1. (i) $x^8+5x^7+5x^6+25x^5+15x^4-5x^3+x^2+4$;
(ii) $2x^8+18x^7+31x^6+19x^5-3x^4+7x^3+21x^2-x-3$.
2. 18981; 103121.
3. $x^{2n+1}+a^{2n+1}$; $x^{2n+1}-2ax^{2n}+2ax^{2n-1}-2ax^{2n-2}+\ldots+2a^{2n}x-a^{2n+1}$.

4. Coefficients: (i) 1, 5, 10, 10, 5, 1; (ii) 1, 3, 6, 7, 6, 3, 1.
   (iii) $a^4+4a^3b+10a^2b^2+12ab^3+9b^4$.

5. $11^2+12^2=133^2-132^2$.

6. (i) $6x^2+15x^{\frac{3}{2}}+38x+52x^{\frac{1}{2}}+32$;
   (ii) $2x^2-3x^{\frac{5}{3}}-x^{\frac{4}{3}}+2x-4x^{\frac{2}{3}}+2x^{\frac{1}{3}}-1$.

Examples 46:

1. Coefficients are 1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1.
   Coefficient is 78.

3. $(1+1)^n=2^n$.

4. (i) $81x^4+108x^3y+54x^2y^2+12xy^3+y^4$;
   (ii) $1-10x+40x^2-80x^3+80x^4-32x^5$;
   (iii) $64a^6+576a^5b+2160a^4b^2+4320a^3b^3+4860a^2b^4+2916ab^5+729b^6$.

5. $1+63x^2+315x^4+189x^6+(7x+105x^3+189x^5+27x^7)\sqrt{3}$.

7. $2+42x^2+70x^4+14x^6$. **8.** $\{(1+x)^3-(1-x)^3\}^2=36x^2+24x^4+4x^6$.

Examples 47:

1. Quotient $3x+4$; Remainder $-6x-13$.

3. (i) x^2+ax+a^2, Remainder $2a^3$;
   (ii) $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$; Remainder $2a^6$.

4. $-\frac{47}{25}x^2+\frac{56}{25}x^3$.

6. $1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+9x^8$.
   Remainder $-10x^9-9x^{10}$.

7. $a^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.

8. (*a*) $1+x+x^2+x^3$; Remainder x^4.
   (*b*) $1+2x+3x^2+4x^3$; Remainder $5x^4-4x^5$.

12. $6x^{\frac{2}{3}}-11x^{\frac{1}{3}}-7$.

Examples 48:

1. $(x+1)(x+4)$. **2.** $x(x-2)(x+3)(x+4)$.

3. $(x+2)(x^2-4x-3)$ or $(x-2\pm\sqrt{7})$; $(x+5)(x+2)(x-2)$.

4. (i) $(x+1)(x-1)(x-6)(x+2)$; (ii) $(x+1)(x-1)(x-2)(x-7)$.

5. (i) $-1, 3, 4$; (ii) $-1, 3, -5$; (iii) $\pm\sqrt{3}, \pm 2$; (iv) $-1, 0, 2, 5$.

6. $3+a$; $(x+1)(x^2+1)$. **7.** $p=2$; $(x-2)^2(x+6)$.

8. (i) 137; (ii) 47. **9.** $af(b)-bf(a)$.

Examples 49:

1. (i) 2; (ii) $-\frac{2}{13}$; (iii) $\frac{3}{2}$. **2.** (i) -15; (ii) -28; (iii) $\frac{15}{28}$.

3. (i) 14; (ii) $1\frac{1}{7}$; (iii) $64\frac{1}{4}$; (iv) $6\frac{7}{9}$.

4. (i) $\frac{5}{6}$; (ii) $-\frac{5}{6}$; (iii) $\frac{62}{105}$; (iv) $\frac{1}{3}$.

5. (i) $x^3+4x^2-20x-48=0$; (ii) $24x^4+14x^2-2x+3=0$.

6. $0, p, -q$. **8.** $y^3+y^2-6y=0$.

9. (i) $(\Sigma a_1)^3-3\Sigma a_1\Sigma a_2a_3+3\Sigma a_1a_2a_3$;
   (ii) $(\Sigma a_1a_2)^2-2\Sigma a_1\Sigma a_1a_2a_3+2\Sigma a_1a_2a_3a_4$.

10. (i) p^2/q^2; (ii) $-(p^3+3q^2)/q^3$.

Examples 50:

1. $a^4+b^4+c^4-2a^2b^2-2a^2c^2+2b^2c^2$, $a^4+b^4+c^4-2a^2b^2-2a^2c^2+2b^2c^2$.

2. (i) $(a+b+c)(a+b-c)$; (ii) $(a-b+c)(a-b-c)$; (iii) $(3a+5b-2c)(3a-5b+2c)$. 3. $3\Sigma a^2-2\Sigma ab$.

4. (i) $a^2+b^2+c^2-2ab-2ac+2bc$; (ii) $a^3-b^3-c^3-3a^2b+3ab^2-3a^2c+3ac^2-3b^2c-3bc^2+6abc$.

5. $a^3+b^3+c^3+2(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)+3abc$. 6. $\frac{1}{4}b^2c^2$.

8. (i) $(a-b+c)(a^2+b^2+c^2+bc-ca+ab)$; (ii) $(a-b-c)(a^2+b^2+c^2-bc+ca+ab)$; (iii) $(a+2b+3c)(a^2+4b^2+9c^2-6bc-3ca-2ab)$; (iv) $(4a-2b+3c)(16a^2+4b^2+9c^2+6bc-12ca+8ab)$.

13. (i) $(b-c)(c-a)(a-b)$; (ii) $-(b-c)(c-a)(a-b)(bc+ca+ab)$.

Examples 51:

1. (i) $(x^2-yz)(x^4+x^2yz+y^2z^2)$; (ii) $(x^2-14)(x^2+1)$; (iii) $(x+\cdot 2)(x-\cdot 2)(x^2+\cdot 04)$; (iv) $(x+3)(x-3)(x^2+9)$; (v) $(x^2+y^2+xy)(x^2+y^2-xy)$; (vi) $\left(x+\frac{1}{x}\right)^2\left(x-\frac{1}{x}\right)^2$; (vii) $(x+a)(x-a)^2$; (viii) $(x^2+2x-4)(x^2-2x-4)$.

2. (i) $(3x-2y+5)(4x+3y-2)$; (ii) $(3x-4y+4)(2x+y-1)$; (iii) $(x-3y-1)(x-y+1)$; (iv) $(x+y+3)(x-y+3)$; (v) $(3x+2y-2)(2x-3y+1)$.

3. (i) $-(b-c)(c-a)(a-b)(a+b+c)$; (ii) $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$; (iii) as for (i); iv) $3(y+z)(z+x)(x+y)$; (v) $5(y-z)(z-x)(x-y)(x^2+y^2+z^2-yz-zx-xy)$.

4. (i) 39; (ii) -15; (iii) $5\frac{5}{16}$; (iv) $\frac{1}{9}$.

5. (i) $-2, -1, 3$; (ii) $-2, -\frac{1}{2}, 2$; (iii) $-6, 1, 1$; (iv) $1, -\frac{1}{2}\pm\frac{1}{2}\sqrt{13}$.

6. (i) $(x-2)(x+2)(2x-1)(2x+1)$; (ii) $(x-1)^2(3x^2+x-1)$; (iii) $(3x+2y-2)(2x-4y+3)$; (iv) $3(a+b+c)(bc+ca+ab)$.

7. $a=7, b=17, c=17$. 9. $(x-a)(x+a)(x+2a)(x^2+xa+a^2)$.

11. $1, -\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$. 12. $5(b+c)(c+a)(a+b)(a^2+b^2+c^2+bc+ca+ab)$.

13. (i) $(b+c)(c+a)(a+b)$; (ii) $(x-4)(2x-9)(2x^2-23x+36)$.

14. $11, -23\frac{2}{3}$. 15. $a=1, b=7, c=6, d=1$.

16. $-4, -3, 2$; $R=a^3p_0+a^2p_1+ap_2+p_3$.

17. $a=\frac{1}{6}(c+4d), b=\frac{1}{6}(6c-d)$ 22. y^5+5y^3+5y.

23. $a=12, b=4$. 24. $2a^7$; 2; $x^6+x^5y+x^4y^2+x^3y^3-x^2y^4-xy^5-y^6$.

CHAPTER VI (pp. 118 to 136)

Examples 52:

1. (i) $\dfrac{2x-17}{(x-3)(x-2)(x+2)}$; (ii) 0.

2. $\dfrac{x^2-3x+4}{(x+2)(x-1)(x-2)}$. 3. $\dfrac{9x^2-3}{(x+1)(2x-1)(x-2)}$.

4. $\dfrac{1}{x^8-1}$. 5. $-\dfrac{2a^3x}{x^6-a^6}$. 6. $\dfrac{2a^4x}{x^6-a^6}$.

7. $\dfrac{x^2-2xz-2yz+3z^2}{(x-y)(y-z)(z-x)}$. 8. $a+b+c$. 12. $\dfrac{2(a^3-1)}{(a-1)^2-b^2}$.

Exercises 53:

1. $\dfrac{1}{x+5}+\dfrac{1}{x+3}$. 2. $\dfrac{1}{x+7}-\dfrac{1}{x+9}$. 3. $\dfrac{2}{3x-1}-\dfrac{1}{x+3}$.

4. $\dfrac{3}{x-5}-\dfrac{2}{x-1}$. 5. $\dfrac{5}{x-2}-\dfrac{4}{x-3}+\dfrac{3}{x-4}$. 6. $\dfrac{1}{x-a}+\dfrac{1}{x+a}$.

7. $\dfrac{3}{x-3}-\dfrac{2}{x-2}$. 8. $1-\dfrac{4}{x-2}+\dfrac{9}{x-3}$. 9. $2+\dfrac{a}{x-a}-\dfrac{a}{x+a}$.

10. $\dfrac{2}{x}-\dfrac{3x+1}{x^2+4}$. 11. $\dfrac{3}{x+1}-\dfrac{x}{x^2+1}$. 12. $\dfrac{x+1}{2(x^2+1)}-\dfrac{1}{2(x+1)}$.

13. $\dfrac{1}{4(x-1)}+\dfrac{1}{(x-1)^2}-\dfrac{1}{4(x+1)}$. 14. $\dfrac{3}{x-1}+\dfrac{2x}{x^2+1}$.

15. $\dfrac{4}{x}-\dfrac{3x+4}{x^2+x+1}$. 16. $\dfrac{x}{x^2-x+1}-\dfrac{1}{x+1}$.

17. $\dfrac{2}{(x+1)^2}+\dfrac{1}{x+3}$. 18. $\dfrac{4}{2x+3}-\dfrac{2}{x+2}-\dfrac{1}{(x+2)^2}$.

19. $\dfrac{1}{1-x}+\dfrac{4}{(1-x)^2}+\dfrac{2+x}{1+x^2}$. 20. $\dfrac{7}{50(x-7)}-\dfrac{7x-1}{50(x^2+1)}$.

21. $\dfrac{2}{2+x^2}-\dfrac{1}{1+x^2}$.

Examples 54:

1. (i) $-\dfrac{1}{4(x+2)}$; (ii) $\dfrac{3}{4}$.

2. $\Sigma\dfrac{a^2}{(a-b)(a-c)}\cdot\dfrac{1}{x-a}$; $\Sigma\dfrac{a^2}{(a-b)(a-c)}\cdot\dfrac{1}{x+a}$.

3. $\dfrac{1}{12(x-1)}-\dfrac{1}{7(x-2)}+\dfrac{1}{16(x-3)}-\dfrac{1}{336(x+5)}$.

4. $\dfrac{4}{x+2}-\dfrac{2}{x-2}+\dfrac{3}{(x-2)^2}-\dfrac{5}{(x-2)^3}$.

5. $\dfrac{1}{8(x-1)}-\dfrac{1}{8(x+1)}-\dfrac{1}{4(x+1)^2}-\dfrac{1}{2(x+1)^3}$.

6. $\dfrac{7}{x}-\dfrac{2}{3-x}-\dfrac{1}{(3-x)^2}$. 7. $1,\ -1,\ -1$. 8. $2,\ -3,\ 0,\ 2$.

9. $-\dfrac{1}{3(1+x)^3}+\dfrac{4}{9(1+x)^2}+\dfrac{8}{27(1+x)}+\dfrac{16}{27(1-2x)}$.

10. 6, 12, 8. **11.** $3y^4 - 37y^3 + 171y^2 - 351y + 275$ where $y = x'$.

12. $2y^3 + 11y^2 + 23y + 17$; $2z^3 - z^2 + 3z - 1$.

13. $-1, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}$. **14.** $\frac{1}{2}, -1, \frac{1}{2}$.

15. $\dfrac{27}{x-3} - \dfrac{11}{x-1} - \dfrac{2}{(x-1)^2} - \dfrac{10}{x-2} - \dfrac{32}{(x-2)^2}$.

16. $\Sigma A \dfrac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}$.

17. $-\dfrac{7x(x-1)(x-2)}{6} + \dfrac{9(x+1)(x-1)(x-2)}{2}$

$-\dfrac{5(x+1)x(x-2)}{2} + \dfrac{(x+1)x(x+1)}{3}$.

18. $\dfrac{1}{2(x-1)^3} + \dfrac{1}{2(x-1)^2} - \dfrac{1}{4(x-1)} + \dfrac{x-1}{4(x^2+1)}$.

19. $\dfrac{1}{(2+x)^4} + \dfrac{4}{3(2+x)^3} + \dfrac{8}{9(2+x)^2} + \dfrac{4}{9(2+x)} + \dfrac{4}{9(1-x)} + \dfrac{4}{9(1-x)^2}$.

20. $b = \dfrac{116}{71}$, $d = \dfrac{114}{71}$, $e = \dfrac{99}{71}$.

Examples 55:

1. (*a*) 15772, 12663; (*b*) 1233322, 1112303.

2. 1001111. **3.** 255 lb.; 128, 32, 4, 1 lb. **4.** 10112, 12002.

6. $140 + 1 + 3 + 27 + 81 = 9 + 243$; 364 lb.;
182 lb. with $1 + 9 + 81$ lb. balances 3 lb., 27 lb. and 243 lb.

7. 34*t*, 1*ete*.

8. The terms give 2000, 3100, 1420, 201; divide repeatedly by 7; 5500 scale of 9.

9. (i) $\cdot 2404\dot{2}$; use $\cdot\dot{4}$ in scale $5 = 1$; (ii) $\cdot 1011$.

10. $\cdot 440140 \ldots$; $\cdot 962643125$.

Examples 56:

2. $\{b^2y^2 - (2bc - apb)y + c^2 - apc + qa^2\}/a^2$. **3.** 3 and 9.

4. (i) $x = y = 0$ or 5; (ii) $x = 2y = 0$ or 1.

Examples 57:

1. $\dfrac{4x^3 + 90x^2 + 550x + 750}{x(x+5)(x+10)(x+15)}$. If $x = 10$, this $= \dfrac{19250}{75000}$.

2. (i) $\dfrac{1}{2(x+1)} - \dfrac{4}{x+2} + \dfrac{9}{2(x+3)}$; (ii) $1 - \dfrac{1}{2(x+1)} + \dfrac{8}{x+2} - \dfrac{27}{2(x+3)}$.

3. (i) $\dfrac{x+3}{x^2 - x + 1} - \dfrac{1}{x+1}$; (ii) $\dfrac{5}{(x+1)^2} - \dfrac{3}{(x+1)^3}$.

4. $\sum_{a,b,c} \dfrac{1}{(a-b)(a-c)(1+ax)}$. **5.** $a = 6, b = 7, c = 1, d = 0$.

6. $3y^3 + 11y^2 + 6y + 3$.

7. (i) $y^4+12y^3+54y^2+108y+93$; (ii) $y^4-12y^3+54y^2-108y+93$.

8. (i) 222132; (ii) $\cdot 75$.

9. 100 lb. $=64+32+4$ lb.; 100 lb. $=81+27-9+1$ lb.

10. $\cdot\dot{2}$, $\cdot\dot{3}6\dot{1}$, $\cdot\dot{1}3\dot{6}$. 11. $xy-2x-3y-24=0$; $y=5$; $x=y=8$ or -3.

TEST PAPERS A (pp. 137 to 143)

A.I:

1. 25. 2. $(x+2y+3z)(x^2+4y^2+9z^2-6yz-3zx-2xy)$.

3. (a) $\frac{1}{2}\pm\frac{1}{2}\sqrt{21}$; (b) $2\cdot 79$, $-1\cdot 79$; $5\frac{1}{2}+\frac{1}{2}\sqrt{21}\simeq 7\cdot 79$.

4. $\frac{31}{20}$, -5, $\frac{3}{10}$, min. $\frac{31}{20}$; $x<1$. 5. $3\cdot 41$, $\cdot 59$.

6. (7, 12) or $(-10, 29)$.

7. $\cdot 1505$, $\cdot 6090$, $2\cdot 2094$, $1\cdot 4313$, $\bar{1}\cdot 6819$; 2 ($\frac{3}{5}$ not possible).

A.II:

1. (a) $2(x-1)(2x-3)$, $(x-2)(x-ax-3-2a)$; (b) $(x-2)(x-3)(x+5)$.

2. (a) 9, (b) 5, 11, (c) $\dfrac{2x-7}{(x-1)(x-2)(x-3)}$.

3. (a) $x^4+2x+\dfrac{1}{x^2}-1$.

4. (a) (3, 1) or $(-5\frac{1}{7}, -5\frac{3}{28})$; (b) $x=4$ (15 unsuitable).

5. $(-1\cdot 5, -4)$, $(3\frac{1}{3}, 8\frac{8}{9})$, $(\frac{1}{4}, -10\frac{1}{8})$. 6. $1\cdot 856$.

7. $0<x<4$; $0<y<4$; $-\cdot 646$, 1, 3, $4\cdot 646$.

A.III:

1. $\dfrac{a}{2x(x^2-\frac{1}{4})}$, $\frac{1}{3}$ hour.

2. (i) 1, $(x-2)$; (ii) $(a+bx-3by/2)(a-bx-by/2)$.

3. (i) 27 or -14; (ii) (3, 2) or $(1\frac{2}{3}, 2\frac{2}{3})$; (iii) 9, 8, ± 3 or 0, -1, 0.

4. (i) $5-x$; (ii) $x-1$. 5. $1\frac{3}{4}$ where $x=\frac{1}{2}$; $1\cdot 62$ or $-\cdot 62$.

6. (i) $1\frac{1}{4}$; (ii) £$582\cdot 4$, £$274\cdot 8$. 7. $\cdot 41$.

A.IV:

1. (i) $x^{\frac{3}{2}}+x^{-\frac{3}{2}}$; (ii) $\cdot 1003$, $1\cdot 107$.

2. (i) $3(x-3)(x^2+3x+9)$, $(x^2+2ax+4a^2)(x^2-2ax+4a^2)$; (ii) 5.

3. (i) $\frac{1}{2}(5\pm\sqrt{41})$; (ii) (2, 3) or $(-\frac{7}{4}, \frac{1}{2})$. 4. (i) 0; (ii) 3.

5. (i) 3. 6. $1\cdot 36$. 7. $-q/p$, r/p.

A.V:

1. (a) $(p-2)(p+2)(p+4)$, $(p+4)(p-2)(p+1)^2$; (b) 3, -24, $(x+4)$.

2. (a) 5; (b) (5, 2) or $(14, -7)$. 3. (ii) 1. 5. $48\cdot 9$ ohms.

6. $\dfrac{3}{2x}-\dfrac{1}{x+1}-\dfrac{1}{2(x+2)}$. 7. $-3a$, 0, b.

A.VI:

1. (i) $(a^2+b^2+c^2+2bc)(a^2+b^2+c^2-2bc)$;
   (ii) $(a^2+b^2+ab\sqrt{2})(a^2+b^2-ab\sqrt{2})$.

2. $5^{-\frac{8}{3}}$, $8^{-\frac{5}{3}}$, $3^{-\frac{5}{8}}$. 3. $p=0$. 4. (i) $-2, 7$; (ii) $-\frac{1}{2}$.

5. (i) $(x+2)(x+1)(x-1)$; (ii) $\dfrac{3}{x-1}-\dfrac{2}{x+2}+\dfrac{5}{2(x+1)}$.

6. £123. 7. $\cdot 69$.

A.VII:

1. (i) $-a$, $-b$; (ii) 9, $8\frac{1}{6}$. 3. $qx^2-p(q+1)x+(q+1)^2=0$.

4. (i) For $x<-3$, $y=-x-1\frac{1}{2}$; for $-3\leqslant x<1\frac{1}{2}$, $y=1\frac{1}{2}$; for $x=1\frac{1}{2}$, $y\geqslant 1\frac{1}{2}$.

4. (ii) $y=\pm x$ and $x^2+y^2=8$. 5. (*b*) $2\frac{1}{2}$.

6. $\frac{5}{9}$; $p=-171$, $q=115$. 7. $-\cdot 077$, $\cdot 192$; 1, $\pm\sqrt{3}$.

A.VIII:

1. (i) $(1, -1)$ or $(-1\frac{23}{26}, \frac{12}{13})$; (ii) $(a-b)(b-c)(c-a)(a+b+c)$.

2. $9x^2+10x+3=0$. 3. $-6, 4, 2, 2\frac{1}{2}$; $-\frac{15}{16}$. 4. $d=\frac{1}{8}\dfrac{l^2w}{T}$.

5. (i) $\dfrac{2}{x-2}-\dfrac{1}{x-1}$; (ii) $\dfrac{4}{x-2}-\dfrac{3}{x-1}-\dfrac{1}{(x-1)^2}$;

   (iii) $1+\dfrac{3}{x-1}+\dfrac{3}{(x-1)^2}+\dfrac{1}{(x-1)^3}$.

6. $\sqrt{6}\simeq 2\cdot 45$. 7. 160501, 2001314.

CHAPTER VII (pp. 144 to 176)

Examples 58:

1. (i) 39; $2n-1$; (ii) 20^2; n^2; (iii) -20; $(-1)^{n-1}n$; (iv) -56; $4-3n$; (v) 21^{21}; $(n+1)^{n+1}$; (vi) $1/2^{12}$; $1/2^{n-8}$.

2. 123. 3. 5, 8, 11, 14; 5, 11, 29, 83.

5. 1, -1, 1, -1, 1; the sum is 1.

6. $6n+6$. 8. 6561. 10. The 7th.

Examples 59:

1. (*a*) (i) 34; (ii) $3n+4$; (iii) 205; (iv) $\frac{1}{2}(3n^2+11n)$.
   (*b*) 63; (ii) $83-2n$; (iii) 720; (iv) $82n-n^2$.

2. 48, 300, 5. 3. 2550, 50. 4. 9.

5. (i) $\frac{1}{2}(9n^2+53n)$; (ii) 9, 14, 19, 24; 1130. 6. 34th; 43.

7. 314, 5140. 8. 13. 10. $n=20$. 11. $\frac{249}{8}$.

12. (i) 150; (ii) a^2+b^2; (iii) $2ab$; (iv) a^3+3ab^2. 13. 17, 30.

16. 2, 4, 6 in. 17. 400 ft. 18. $\frac{1}{4}(n^2+13n)$.

Examples 60:

1. (i) 2; (a) $5 \cdot 2^{29}$, (b) $5 \cdot 2^{n-1}$, (c) $5(2^{30}-1)$, (d) $5(2^n-1)$;
(ii) $\frac{2}{3}$; (a) $2^{32}/3^{27}$, (b) $2^{n+2}/3^{n-3}$, (c) $216\{1-(\frac{2}{3})^{30}\}$, (d) $216\{1-(\frac{2}{3})^n\}$;
(iii) $\frac{3}{5}$; (a) $(\frac{3}{5})^{29}$, (b) $(\frac{3}{5})^{n-1}$, (c) $\frac{5}{2}\{1-(\frac{3}{5})^{30}\}$, (d) $\frac{5}{2}\{1-(\frac{3}{5})^n\}$.
2. (i) 18; (ii) 1·1; (iii) ar^2; (iv) x^2-y^2.
3. (i) $48\left\{\left(\frac{5}{4}\right)^n - 1\right\}$; (ii) $\dfrac{x^{3n}-1}{x(x^3-1)}$; (iii) $\dfrac{ax(x^{3n}-1)}{x^3-1}$.
(iv) $\dfrac{2x\left\{1-\left(\dfrac{(-x)^n}{2^n}\right)\right\}}{2+x}$.
4. 36 or -36. 5. $\frac{256}{81}$ or $-\frac{256}{81}$. 6. $2\{1-(\frac{1}{2})^n\}$; $\frac{2}{3}\{1-(-\frac{1}{2})^n\}$.
8. (i) $1{\cdot}056\times10^{-19}$; (ii) $3\frac{1}{3}\times10^{-40}$; (iii) $3{\cdot}105\times10^{-10}$.
12. Same statement. 13. 2. 14. 6, 12, 24, ...
17. *AD*, *AB*, *CA*.

Examples 61:

1. $\frac{1}{8}$. 2. $-\frac{12}{5}$. 7. (i) $6\frac{1}{2}$, 6, $5\frac{7}{13}$; (ii) p, $\sqrt{(p^2-q^2)}$, $\dfrac{p^2-q^2}{p}$.

Examples 62:

1. $2^{100}/10^{49} \simeq 1{\cdot}27\times10^{-19}$, $5{\cdot}44\times10^{-20}$. 2. (i) 22; (ii) 7.
3. $3{\cdot}25\times10^{18}$; $8{\cdot}5\times10^{-8}$.
4. $32\left(1-\dfrac{1}{2^{15}}\right)$; (i) 32; (ii) 1/1024. 5. $5{\cdot}9\times10^{38}$; 162.
6. They are approximately $6{\cdot}3\times10^{11}$ and $3{\cdot}8\times10^{12}$.
7. (i) $\frac{13}{90}$; (ii) $\frac{203}{999}$; (iii) $\frac{403}{990}$.
8. 11 terms; no, the sum to infinity is 2·5. 9. $\dfrac{8}{3}\left\{1-\dfrac{1}{2^{2n}}\right\}$; $\frac{8}{3}$.
11. $1{\cdot}84\times10^{19}$. 12. $\dfrac{ax^{100}(1-x^{50})}{1-x}$; $\dfrac{ax^{150}}{1-x}$.

Examples 63:

1. £655; £19. 2. 3·1%. 3. £3684.
4. $2400\times1{\cdot}05^n$, 3·4014, ·0212, 3·3802 + ·0212*n*.

Examples 64:

1. 7·442, 8·930. 2. 7·84, 8·58.

Examples 65:

3. $2xy/(x+y)$. 4. $\frac{1}{2}(x+y)$; $(px+qy)/(p+q)$. 7. $\frac{1}{2}$ in.
8. $5\frac{5}{11}$. 9. 25.

Examples 66:

1. 5, 8, 11, 14, 17, 20. 2. 4, 8, 16, 32, 64. 3. $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}$.
4. $\frac{1}{6}(5a+b)$, $\frac{1}{6}(4a+2b)$, $\frac{1}{6}(3a+3b)$, $\frac{1}{6}(2a+4b)$, $\frac{1}{6}(a+5b)$.
5. a^2b, ab^2. 6. $\frac{30}{9}$, $\frac{30}{8}$, $\frac{30}{7}$.

Examples 67:

1. 5′ 8·424″. 2. 2s. $11\frac{2}{5}$d. 3. 13 yr. 7·68 m. 4. 47·4, 47·5.
5. 5 ft. 7·3 in.

Examples 68:

4. $a+(n-1)d,\ a+nd,\ a+(n+1)d,\ a+(n+2)d$. 5. $n=13$.
7. G.M. $=1-x^4$; $1 \pm 2x + 2x^2 \pm 2x^3 + x^4$.
10. (i) $\frac{23}{99}\left(1-\frac{1}{10^{2n}}\right)$; (ii) $\frac{23}{99}$ (iii) $\{n \times 23 - \cdot 2323 \ldots (2n \text{ places})\}/99$.
11. $p^2/(p-q)$. 12. 4 : 1. 13. Any one with $a=d$.
16. $A \cdot \left(\frac{100-r}{100}\right)^n$; (i) £835 16s.; (ii) 11 yr. (10·85).
17. $\frac{1}{2}n(n-1)b - \frac{1}{2}n(n-3)a$; (i) 11369.
18. $n(2n+1) - 2n(n+1) = -n$.
19. $(a+b+c)uvw/(avw+bwa+cuv)$. 20. $2 \pm \sqrt{3} : 1$.
21. 189·4 yd. 22. 51·25. 23. $6\{1-(\frac{2}{3})^n\}$; 16.
24. $166\frac{1}{4}$; 18. 25. $6r-3$; 3; 6417.
26. $279/28(1-\sqrt[3]{4/7})$; 184th.
27. $\frac{a^2}{a-b}\left\{1-\left(\frac{b}{a}\right)^n\right\}$; 24; 88·6 degrees. 28. £136.
29. £1099. 30. $2^n + n^2 + n - 1$; 1133.
31. $a(1-x^n)/(1-x)$; £175. 33. 22. 34. $(1-x^n)/(1-x)$.
35. 630π. 36. 2430.
37. $A(R^n-1)/R^n(R-1)$ where $R = 1 + r/100$; £1623.
38. 9217. 40. 65%; 59.

CHAPTER VIII (pp. 177 to 209)

Examples 69:

1. (i) $5! = 120$; (ii) $6! = 720$; (iii) $7! = 5040$; (iv) $9! = 362880$.
2. $6! = 720$. 3. (i) $\frac{1}{30}$; (ii) 132; (iii) $n(n-1)$; (iv) $(n+1)n$.
4. $11! = 39{,}916{,}800$. 5. 96.
6. (i) 9; (ii) $n(n-1)(n-2)$; (iii) $(n-3)!$; (iv) 8064; (v) 2·1; (vi) ·0045.

Examples 70:

1. (i) 360; (ii) 180; (iii) 30240; (iv) 34650. 2. 792.
3. 27720.

Examples 71:

1. Total = 243. 2. $2^{10} - 1 = 1023$. 3. 42. 4. 20.
5. 32, 62. 6. $3^{12} = 531{,}441$.

Examples 72:

2. $n(n-1)(n-2)(n-3)(n-4)=n!/(n-5)!$; $n-6$. 3. ${}_{20}P_3$ by 360.
4. $10!/7!$; $5!$; $20!/13!$; $18!/9!$. 6. 4. 7. 6720; 600.
8. $10!\,6!\,5!\,5!\,4!$. 9. $28 \cdot (29!)$. 10. 1680, 840, 1440. 11. 36, 48.

Examples 73:

1. Combinations *abc*, *abd*, *acd*, *bcd*; Permutations of *bcd* are *bcd*, *bdc*, *cdb*, *cbd*, *dbc*, *dcb*.
3. 190, 120, 55, 364. 4. (i) ${}_{14}C_3=364$; (ii) ${}_6C_3=20$.
5. (i) 56; (ii) 25; (iii) 10.
6. (i) 7, 21, 35, 35, 21, 7; (ii) 8, 28, 56, 70, 56, 28, 8; one if n is even, 2 if n is odd.
7. ${}_nC_3$; 455. 9. Put $r=2$, ${}_nC_1=n$. 10. ${}_9C_5=126$.
12. (i) 18; (ii) 16 or 5. 13. ${}_{15}C_3\times{}_{30}C_3=1847300$.
14. ${}_{14}C_2\cdot{}_{30}C_3+{}_{15}C_3\cdot{}_{29}C_2+{}_{14}C_3\cdot{}_{29}C_3$. 15. 1961256.

Examples 74:

2. $2\binom{n}{r}$. 3. $32=2^5$. 4. 32. 6. 11.
8. There are 21 ways of choosing 5 in block P and only 11 of these are covered.
9. 90.

Examples 75:

2. 21600. 3. 720. 4. (i) 60; (ii) 120; (iii) 20; (vi) 325.
5. 6, 6. 6. 2, 4, 8, 16. 7. 2880, 576. 8. (i) $19!$; (ii) $18 \cdot 19!$.
9. 9. 10. (ii) 136; (iii) 151.

Examples 76:

1. (i) $16!$; (ii) $12 \cdot 11 \cdot 14!$. 2. 10^5. 3. 179. 4. 53,284.
5. (i) 5880; (ii) 1560. 6. $n!-2(n-1)!$. 7. 10. 8. 60.
9. $\frac{1}{2}n(n+1)$. 10. 6720. 11. 10. 12. 10640.
13. $\frac{1}{2}n(n-3)$; $\frac{1}{6}(n^3-9n^2+20n)$. 14. (*a*) 192; (*b*) 72000.
15. Red balls alike, white balls alike (i) 175; (ii) 2.
16. $n!/n_1!\,n_2!\,n_3!\,\ldots$.

Examples 77:

1. (i) $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$;
(ii) $2x^6+120x^4+480x^2+128$;
(iii) $32a^5-240a^4b+720a^3b^2-1080a^2b^3+810ab^4-243b^5$.
2. (i) $16x^4-32x^3y+24x^2y^2-8xy^3+y^4$;
(ii) $24x^2+216/x^2$; (iii) $x^8+4x^5+6x^2+4/x+1/x^4$.
3. (i) $220x^3a^9$; (ii) $-15360x^3$; (iii) $22680x^3a^4$;
(iv) $-30x^3$; (v) $-61236x^3$; (vi) $2016x^3$.

4. (ii) $-20x^4$; (iii) $175x^4$; (iv) $5376x^4$. 5. ${}_7C_2 \cdot 2^2 \cdot 3^5x^2y^5 = 20412x^2y^5$.
6. (i) $-80a^4$; (ii) $-10a^7$. 7. (i) 3; (ii) 46. 8. $280x^3b^4/81$.
9. $231x^6y^6/16$. 10. 70. 11. $\frac{4641}{1024}$. 12. $2x^6+60x^4+120x^2+16$.
15. $n=8$. 17. (i) 0; (ii) 243; (iii) 1.

Examples 78:

1. $\dfrac{21-n}{3n}$. 2. $1+\dfrac{18}{5}+\dfrac{18 \cdot 17}{2 \cdot 5^2}+\dfrac{18 \cdot 17 \cdot 16}{6 \cdot 5^3}$.
3. (i) 5th; (ii) 4th. 4. 5th.
5. (i) ${}_{18}C_7 \cdot 3^7 \cdot 5^{11}$; (ii) 16th term; ${}_{18}C_3 \cdot 4^{15}$.
6. (i) 5th; (ii) 2nd; (iii) 3rd.
7. (i) Of 3rd or 4th terms (equal); (ii) of $(n+1)$th term.
8. $(n+1)$th; 9th. 10. $\frac{9}{4}<|x|<3$.

Examples 79:

1. 1·0936, 1·06165, 142·0448; tables give 1·093, 1·059, 142·1.
2. By ·000003. 4. $1+\dfrac{r}{100}+\dfrac{51r^2}{1040000}$. 5. ·868643.

Examples 80:

1. $1+x+x^2+x^3+\dots$; $1+2x+3x^2+4x^3+\dots$;

$$1+3x+\frac{3 \cdot 4}{1 \cdot 2}x^2+\frac{4 \cdot 5}{1 \cdot 2}x^3+\dots.$$

2. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4-\frac{7}{256}x^5-\dots$.
3. (i) $1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3$; (ii) $1+\frac{4}{3}x+\frac{20}{9}x^2+\frac{320}{81}x^3$.
4. (i) $\dfrac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^8x^6}{3^6 \cdot 6!}$; (ii) $\dfrac{11!}{5!\,6!}x^5$.
5. (i) $(-1)^{r-1}\dfrac{5 \cdot 3 \cdot 7 \cdot 11 \dots (4r-13)}{(r-1)!\,4^{r-1}}x^{r-1}$;

(ii) $p(p+1)(p+2)\dots(p+r-1)x^r/r!$;

(iii) $p(p+q)(p+2q)\dots\{p+(n-1)q\}x^n/q^n \cdot n!$.

6. (i) $10{\cdot}0498 \simeq 10{\cdot}050$; (ii) 10·01; (−·00001); (iii) 1·995.
7. 1 − first 4 terms of series to be summed.
8. $1-2\left\{\dfrac{x}{2}+\dfrac{x^2}{4}+\dfrac{x^3}{4}+\dfrac{5x^4}{16}\right\}$; $1-2\left\{\dfrac{x}{2}+\dfrac{x^2}{4}+\dfrac{x^3}{8}+\dfrac{x^4}{16}\right\}$.

Series differ by $\dfrac{2}{10^6}-\dfrac{8}{10^8}$; $\sqrt{6}$ and $\dfrac{485}{198}$ are $\dfrac{5}{2}\times$ the series.

Examples 81:

1. ·1%, 39·11 in., 1·8 min. 4. In excess. 5. Increase ·2%.
6. ·6%.

Examples 82:

2. $\frac{21}{128}$. 3. (i) $\frac{1}{36}$; (ii) $\frac{5}{72}$; $\dfrac{n(n-1)}{72} \cdot \left(\dfrac{5}{6}\right)^{n-2}$.

4. $\frac{5}{39}$. 5. $\frac{7}{52}$. 6. $\frac{7}{26}$.
7. 3 ways for 10, 6 ways for 7. 8. (i) $1/3^{10}$; (ii) $20/3^8$.
9. $\frac{1}{6}$. 10. $\frac{931}{992}$.

Examples 83:

1. (i) $\frac{1}{3}$; (ii) $\frac{1}{2}$. 2. (i) $\frac{2}{13}$; (ii) $\frac{33}{221}$.
3. (i) $\frac{5}{33}$; (ii) $\frac{7}{22}$; (iii) $\frac{35}{66}$. 4. (i) $\frac{81}{361}$; (ii) $\frac{1}{19}$; (iii) $\frac{9}{361}$.
5. (i) $\frac{7}{25}$; (ii) $\frac{3}{100}$.

Examples 84:

1. $\frac{1}{9}$. 2. (i) $20/9^4$; (ii) $375/9^4$. 3. $\frac{6}{11}$.
4. ${}_8C_3 \cdot 5^3/6^8$. 5. $\frac{32}{81}$. 6. $\frac{1}{4}$. 7. $\frac{7}{72}$.

Examples 85:

1. (i) 8. 2. (i) 42, 840; (ii) 30, 360; (iii) 56, 1680.
3. $8!\,.\,7!$, $5\,.\,8!\,7!$. 4. ${}_{15}C_5 \times {}_{12}C_3 = 660660$.
5. ${}_{11}C_5 \cdot a^6/b^5$. 6. ${}_9C_4 \cdot 2^5 3^4$. 7. 27, ${}_{12}C_4 \cdot 3^8$.
8. (ii) $(n-3)/4$. 9. ${}_{16}C_8 = 12870$. 10. ${}_nC_3 a^{n-3} b^3$, 19.
11. (ii) 190, use $(1+x)^{20}/x^{10}$.
12. $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4$; $-\frac{(3n-4)x}{3n}$; $1-\left(\frac{1}{3}\right)^{\frac{1}{3}} \simeq \cdot 3066$.
14. $\frac{1}{360}$. 15. $\frac{1}{16}$.

CHAPTER IX (pp. 210 to 226)

Examples 86:

1. $2n$. 2. $2(n-1)$. 3. $4n-1$. 4. $3n(n+1)$.
5. $6n(n+2)$. 6. $4n(n+1)(n+2)$; in Nos. 1 to 6, $s_1 = u_1$.
8. (i) 3080; $\frac{1}{3}n(n+1)(n+2)$; (ii) 3290; $\frac{1}{6}n(n+1)(2n+7)$;
(iii) 12740; $\frac{1}{3}n(n+1)(4n+11)$;
(iv) 14100; $\frac{1}{3}n(2n+5)(2n+7)$;
(v) 56210; $\frac{1}{12}n(n+1)(n+2)(3n+13)$;
(vi) 59290; $\frac{1}{12}n(n+1)(n+2)(3n+17)$.
9. (i) 42,925; (ii) 85,320; (iii) 171,700; (iv) 112,761.
10. 216,225; (i) 1,729,800; (ii) 5,838,075.
11. (i) $\frac{1}{12}n(n+1)(n+2)(3n+5)$; (ii) $\{\frac{1}{4}(n-1)n\}^2$.
12. 107,745. 13. $\frac{1}{4}(n+1)(n+2)(n+3)(n+4)-6$.
14. (i) $\frac{1}{12}\{(3n-2)(3n+1)(3n+4)(3n+7)+56\}$.
(ii) $\frac{1}{10}2n(2n+2)(2n+4)(2n+6)(2n+8)$.
15. $1-\frac{1}{n+1}$; limiting sum $=1$.
16. (i) $\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3n+4}\right)$; $\frac{1}{12}$. (ii) $\frac{1}{4}\left(\frac{1}{3}-\frac{1}{4n+3}\right)$; $\frac{1}{12}$.
(iii) $\frac{1}{4}\left(\frac{1}{5}-\frac{1}{4n+5}\right)$; $\frac{1}{20}$. (iv) $\frac{1}{2}\left\{\frac{1}{6}-\frac{1}{(n+2)(n+3)}\right\}$; $\frac{1}{12}$.

Examples 87:

1. $\dfrac{1}{4}-\dfrac{1}{2(n+1)(n+2)}$. 2. $\dfrac{3}{4}-\dfrac{1}{2}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}\right)$.

3. $\dfrac{1}{168}-\dfrac{1}{6}\dfrac{1}{(3n+4)(3n+7)}$. 4. $\dfrac{1}{4}\left\{\dfrac{1}{80}+\dfrac{1}{99}-\dfrac{1}{(n-1)(n+1)}-\dfrac{1}{n(n+2)}\right\}$.

5. $\dfrac{1}{2}\left\{\dfrac{1}{2}-\dfrac{1}{(n+1)(n+2)}\right\}-\dfrac{1}{3}\left\{\dfrac{1}{6}-\dfrac{1}{(n+1)(n+2)(n+3)}\right\}$.

6. $a=3,\ b=1$; $\dfrac{4}{3}-\dfrac{3n+4}{(2n+1)(2n+3)}$.

7. $1-\dfrac{1}{(n+1)!}$. 8. $\frac{1}{4}, \frac{3}{4}, \frac{1}{168}, \frac{179}{31680}, \frac{7}{36}, \frac{4}{3}, 1$.

9. (i) $u_n=\dfrac{3}{(2n+1)(2n+3)}$; (ii) $u_n=\dfrac{2}{(5n+2)(5n-3)}$.

10. $\dfrac{-119}{5!}$; $-1+\dfrac{1}{(n+1)!}$.

Examples 88:

6. $p+1,\ \frac{1}{2}(p+1)(p+2),\ \frac{1}{6}(p+1)(p+2)(p+3),\ \ldots$

$$\frac{1}{r!}(p+1)(p+2)\ldots(p+r).$$

8. Use $x^{n+1}-a^{n+1}=x(x^n-a^n)+a^n(x-a)$.

Examples 89:

1. $\{1-(n+1)x^n+nx^{n+1}\}/(1-x)^2$.
2. $\{1+x-(2n+1)x^n+(2n-1)x^{n+1}\}/(1-x)^2$.
3. $\{1+(-1)^{n+1}(n+1)x^n+(-1)^{n+1}nx^{n+1}\}/(1+x)^2$.
4. $\dfrac{9}{4}\left\{1-\dfrac{n+1}{3^n}+\dfrac{n}{3^{n+1}}\right\}$. 5. $\dfrac{5}{12}+\left(\dfrac{1}{12}+\dfrac{n}{2}\right)\left(-\dfrac{1}{5}\right)^{n-1}$.
6. $\{1+x-x^n(n+1)^2+x^{n+1}(2n^2+2n-1)-x^{n+2}\,.\,n^2\}/(1-x)^3$.
7. (i) Write $-x^2$ for x in answer to No. 6;
   (ii) $(1+x)^4S_n$
   $$=1-4x+x^2+(-1)^{n+1}\{(n+1)^3x^n+3(n^3+2n^2+4)x^{n+1}$$
   $$+(3n^3+3n^2-3n+1)x^{n+2}+n^3x^{n+3}\}.$$
8. Subtract $\dfrac{x-x^n}{1-x}$ from answer to No. 6.
9. $(1-x)^3S_n=2x-(n^2+n)x^n+(2n^2-2)x^{n+1}-(n^2-n)x^{n+2}$.
10. $(1-x)^3S_n=2-x-x^2-(n^2+4n+2)x^n$
   $$+(2n^2+6n-1)x^{n+1}-(n^2+2n-1)x^{n+2}.$$

Examples 90:

1. $40u_2-39u_2$. 2. $205u_2+204u_1$. 3. 5, 7, 11, 19.
4. $2\,.\,3^n+(-1)^n\,.\,4^n$. 5. $6\,.\,2^r-5$.
6. $u_n=2\,.\,3^n-4^n$; $S_n=3(3^n-1)-\frac{4}{3}(4^n-1)$.
7. $\{x-(-1)^r[(2^{r+1}-1)x^{r+1}+(2^{r+1}-2)x^{r+2}]\}/(1+3x+2x^2)$.

8. Scale is $1 - x - 2x^2$;

$$(1 - x - 2x^2) S_n = 1 + 7x - 3 \cdot 2^n x^n - \{3 \cdot 2^n + 4(-1)^n\} x^{n+1}.$$

9. $u_r - 5u_{r-1} + 6u_{r-2} = 0$; $0 + 6 + 30 + 114 + \dots$;

$$\frac{6x}{1 - 5x + 6x^2} = \frac{6}{1 - 3x} - \frac{6}{1 - 2x}.$$

10. $2 + 3(-2)^r - 2^r$; scale is $1 - x - 4x^2 + 4x^3$;

Generating function, $\dfrac{4 - 10x}{(1 - x)(1 - 4x^2)}$.

Examples 91:

(i) $6 + 8 + 10 + 12 + \dots + (2r + 4) + \dots$;

(ii) $\frac{1}{2} - \frac{1}{3} - \frac{1}{12} - \frac{1}{30} - \dots - \dfrac{2}{(r-1)r(r+1)} - \dots$;

(iii) $1 + 7 + 19 + 37 + \dots + (3r^2 - 3r + 1) + \dots$.

2. (i) 3025; $55^2 - 45^2$; (ii) $\{\frac{1}{2}n(n+1)\}^2 - \{\frac{1}{2}(n-1)n\}^2$.

3. $66^2 - 15^2$. **4.** $(n-2)^3 + (n-1)^3 + n^3 + (n+1)^3 + (n+2)^3$; $8^3 + 9^3$.

5. (i) $\left[1 + \frac{1}{2} + \frac{1}{3} - \left(\dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3}\right)\right] \div 3$;

(ii) $\dfrac{1}{4}\left\{\dfrac{1}{15} - \dfrac{1}{(2n+3)(2n+5)}\right\}$.

6. $(n+1)! - 1$. **7.** $1 - \dfrac{1}{20!}$. **8.** 20100.

9. $6r - 3$; 3; $4100 + 2842 - 525 = 6417$. **10.** $\frac{1}{2}n^2(n+1)$.

11. $\frac{1}{3}, \frac{3}{2}, 1, 0$; $3 - 2^{n+1} + (2n-1)2^n$.

12. $-3(2n-1)$, $3n^2 - 3n + 1$; $n(n+1)(2n+1)/6$.

13. 627,500. **14.** n^3. **15.** $6n^2$.

16. $\frac{1}{2}, -\frac{1}{2}$. **17.** $\frac{1}{8}(2n-1)(2n+1)(2n+3)(2n+5)$.

19. $\frac{1}{6}, -\frac{1}{12}$; 2, 2. **20.** 1, 8, 8. **21.** $\frac{1}{6}$.

22. (i) $5 \cdot 2^n + 3(-3)^n$; (ii) $5 \cdot 3^n + 7(-2)^n$.

23. $\{u_1 + (u_2 - u_1)x - (u_n + u_{n-1})x^{n+1} - u_n x^{n+2}\}/(1 - x - x^2)$.

24. $\frac{9}{4}n^2(n+1)(3n-1) - n$. **25.** $(7 \cdot 5^{n-1} - 1)/2$.

CHAPTER X (pp. 227 to 255)

Examples 92

2. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$.

3. $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{2} \cdot \dfrac{x^3}{2^3} - \dfrac{1 \cdot 3 \cdot 5}{4!} \dfrac{x^4}{2^4} + \dots$

$$+ (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{n!} \cdot \frac{x^n}{2^n}.$$

6. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

7. $\dfrac{2}{3}(1 + x)^{\frac{3}{2}} = \dfrac{2}{3} + x + \dfrac{1}{2} \cdot \dfrac{x^2}{2} - \dfrac{1}{8} \cdot \dfrac{x^3}{3} + \dfrac{1}{16} \cdot \dfrac{x^4}{4} - \dfrac{5}{128} \cdot \dfrac{x^5}{5} + \dots$.

8. (i) $n \cdot 2^{n-1}$; (ii) $\dfrac{1}{n+1}(2^{n+1}-1)$.

10. $1+6x+21x^2+56x^3+126x^4$.

11. $1+\dfrac{9x}{2}+\dfrac{99x^2}{8}+\dfrac{429x^3}{16}+\dfrac{6435x^4}{128}$; after the 9th term.

12. $1+ny+\dfrac{n(n+1)}{2!}y^2+\dfrac{n(n+1)(n+2)}{3!}y^3$.

13. $(-1)^r\dfrac{(r+1)(r+2)}{2}x^r$.

15. $1+px+\dfrac{p(p+q)}{2!}x^2+\dfrac{p(p+q)(p+2q)}{3!}x^3+\dots$

$$+\frac{p(p+q)\dots(p+nq-q)}{n!}x^n+\dots .$$

16. $1+\frac{1}{2}x+\frac{3}{8}x^2-\frac{3}{16}x^3+\dots$.

Examples 93:

1. $3-3x+9x^2$; $1+(-1)^r 2^{r+1}$; $|x|<\frac{1}{2}$.
2. $3+x+3x^2$; $2+(-1)^r$; $|x|<1$.
3. $-1+16x-40x^2$; $2^{r+1}+(-1)^{r+1}\cdot 3\cdot 4^r$; $|x|<\frac{1}{4}$.
4. $\dfrac{1}{6}+\dfrac{19x}{36}+\dfrac{211x^2}{216}$; $\dfrac{3^r}{2^{r+1}}-\dfrac{2^r}{3^{r+1}}$; $|x|<\dfrac{2}{3}$.
5. $1+2x+2x^3$; $1+(-1)^{r-1}$; $|x|<1$.
6. $(a+b)+(2a-b)x+(4a+b)x^2$; $a\cdot 2^r+b(-1)^r$; $|x|<\frac{1}{2}$.
7. 3·0366. 8. $1+\frac{1}{4}x^2+\frac{5}{32}x^4+\dots$; 0·33437.
9. $(1-x)^{-2}$; ·99.
12. See Nos. 1, 3; $a_n+a_{n-1}-2a_{n-2}=0$; $a_n+2a_{n-1}-8a_{n-2}=0$.
13. $(r+1)b_{r+1}+2nb_r-(r-1)b_{r-1}=0$.
14. $2^{n+1}-3^n/2^{n+1}$; $|x|<\frac{1}{2}$. 15. $|x|<\frac{1}{2}$; $|x|<2$.

Examples 94:

1. $1+2x+2x^2+\frac{4}{3}x^3$; $\dfrac{2^n x^n}{n!}$.
2. $1-3x+\dfrac{9x^2}{2}-\dfrac{9x^3}{2}$; $\dfrac{(-1)^n 3^n x^n}{n!}$.
3. $1-\dfrac{p}{q}+\dfrac{p^2}{2q^2}-\dfrac{p^3}{6q^3}$; $\dfrac{(-1)^n p^n}{n!\,q^n}$.
4. $2+1+\frac{1}{12}+\frac{1}{360}$; $\dfrac{2}{(2n)!}$.
5. $1+2x+\frac{3}{2}x^2+\frac{2}{3}x^3$: $\dfrac{x^n(n+1)}{n!}$.
6. $1-3x+\frac{5}{2}x^2-\frac{7}{6}x^3$; $\dfrac{(-1)^n(2n+1)x^n}{n!}$.
7. 2·7183. 8. 1·105,170,918,076. 9. 3·762.

10. $(\log_e 2)^{n-1}/(n-1)!$; 2.

11. (i) $1+\frac{1}{2}x-\frac{7}{8}x^2-\frac{23}{48}x^3$; $\dfrac{1+4r-4r^2}{2^r \,.\, r!}$.

(ii) $4+4x+3x^3+\frac{5}{3}x^3$; $\dfrac{2^r+2}{r!}\,x^r$.

12. $p+(3p+q)x+\dfrac{(9p+6q)}{2}x^2+\dfrac{9(p+q)}{2}x^3$; $\dfrac{p \cdot 3^r}{r!}+\dfrac{q \cdot 3^{r-1}}{(r-1)!}$.

13. 10th. **14.** 23rd.

Examples 95:

1. (i) $\cdot 02-\dfrac{\cdot 0004}{2}+\dfrac{\cdot 000008}{3}-\ldots$;

(ii) $-\left\{\cdot 01+\dfrac{\cdot 0001}{2}+\dfrac{\cdot 000001}{3}+\ldots\right\}$;

(iii) $\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{24}x^6+\ldots$.

2. (i) 1·0986 ; (ii) ·2231, 1·6094.

3. 2·3025 [2·3026] ; $\log_{10} 2=\log_e 2 \div \log_e 10=\cdot 301$.

4. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}$. **5.** $2x-2x^2+\dfrac{8x^3}{3}-\ldots+(-1)^{n-1}\dfrac{2^n x^n}{n}+\ldots$.

6. $2x-x^2+\dfrac{2x^3}{3}-\ldots+(-1)^{n-1}\dfrac{2 \,.\, x^n}{n}+\ldots$.

7. $x^2-\dfrac{x^4}{2}+\dfrac{x^6}{3}-\ldots+(-1)^{n-1}\dfrac{x^{2n}}{n}+\ldots$.

8. $-x-3x^2-\dfrac{7x^3}{3}-\ldots-\dfrac{x^n}{n}\left\{2^n+(-1)^n\right\}+\ldots$.

9. $4x-5x^2+\dfrac{28x^3}{3}-\ldots+(-1)^{n-1}\dfrac{x^n}{n}(3^n+1)+\ldots$.

10. ·7178. **13.** ·09531.

14. 0·6931, 1·0986, 1·6094, 2·3026 are the logs. to base *e*.

Examples 96:

1. $1-x^2+\frac{1}{3}x^4-\frac{2}{45}x^6+\ldots$.

2. (i) ·099,833,42 ; (ii) ·5403. **3.** $2x-\dfrac{2^3x^3}{3!}+\dfrac{2^5x^5}{5!}-\dfrac{2^7x^7}{7!}$; same

4. Five terms give $\dfrac{\sin \pi}{\pi} \simeq -\cdot 004$.

5. $x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots$.

6. 3rd term $\simeq \cdot 000003$; 2 terms give ·1045.

10. $\sin^{-1} x=x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1 \cdot 3}{2 \cdot 4}\cdot\dfrac{x^5}{5}+\dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\dfrac{x^7}{7}+\ldots$.

11. $e^{-t}\sin t = t - t^2 + \dfrac{t^3}{3} - \dfrac{t^5}{30} + \dots$.

12. $16\left\{\dfrac{1}{5} - \dfrac{1}{3}\left(\dfrac{1}{5}\right)^3 + \dfrac{1}{5}\cdot\left(\dfrac{1}{5}\right)^5\right\} - 4\left\{\dfrac{1}{239} - \dfrac{1}{3}\left(\dfrac{1}{239}\right)^3 + \dots\right\}$.

13. $\tan\theta \simeq \theta + \dfrac{\theta^3}{3} + \dfrac{2\theta^5}{15}$.

Examples 97:

4. $1 + x\log_e a + \dfrac{x^2}{2!}(\log_e a)^2 + \dfrac{x^3}{3!}(\log_e a)^3 + \dots$.

5. $x - x^2 + \dfrac{1}{3}x^3 + \dfrac{1}{30}x^5 + \dots$.

6. L.H.S. $= 1 + 2x + x^2 - \dfrac{x^4}{6} + \dots$.

TEST PAPERS B (pp. 256 to 262)

B.I:

1. 161. 2. £4670 (£4650 with 4-figure tables).
3. 3 figs. (·994793). 4. 216; $\frac{37}{65}$. 6. Less.

B.II:

1. $x = \pm\frac{4}{3}$, $y = \mp\frac{2}{3}$ or $x = \pm\frac{2}{3}$, $y = \mp\frac{4}{3}$. 2. (ii) 11.
4. (i) -4, -4, 5; (ii) $(x-1)^2(x-3)$. 5. $\frac{1}{36}$.

B.III:

1. (i) (0, 2), $(-3, 4)$; (ii) 1734. 2. 13.
4. $\dfrac{3}{4} - \dfrac{1}{2}\left(\dfrac{1}{n+1} + \dfrac{1}{n+2}\right)$; $\dfrac{3}{4}$. 5. 66. 6. 1, $\frac{1}{8}$.

B.IV:

1. Yes. 2. ·807.
4. $\dfrac{1}{x-1} + \dfrac{1}{(x-1)^2} + \dfrac{1}{x+1} + \dfrac{x+1}{x^2+1}$; $2n(2n-1)$.
5. $-n(2n+1)$; $(2n-1)/(2n+2)$.
6. (ii) $\frac{1}{8}(6S_1 + 4S_2)$.

B.V:

1. A.M., H.M.; the first.
2. (i) $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.
2. (ii) $(2x+y-1)(x-y+2)$. 3. $x \simeq 8{\cdot}25$; $1{\cdot}3222$.
4. (i) -3, $-\frac{16}{7}$; (ii) $(-1, 11)$, $(\frac{3}{2}, \frac{7}{2})$. 5. 5.
6. $x = y + \dfrac{y^2}{2} + \dfrac{y^3}{6} + \dfrac{y^4}{24}$.

B.VI:

1. £$P(1-1/q)^n$. 3. (i) $-2, 5$; (ii) $3, 6$.
4. 1, 6, 15, 20, 15, 6, 1; 40 : 3. 5. 0·45, 3·52.

B.VII:

4. $n=-1$ (not a positive integer).

B.VIII:

1. $2, 2, -3, -2$.
2. (a) $x=\frac{1}{2}a(1\pm\sqrt{3}),\ y=\frac{1}{2}a(1\mp\sqrt{3})$;
 (b) $x=a(c-a)/(c-b),\ y=b(c-b)/(c-a)$.
3. Exclude $-3\leqslant r\leqslant -2$ and $1\leqslant r\leqslant 6$. 4. $-3, 3, -1$.
5. 1·2213. 6. (i) $n(n+1)(2n+1)/6$.
6. (ii) $n(n+1)^2(n+2)/12$; $1+(x+x^2)e^x$.

CHAPTER XI (pp. 263 to 285)

Examples 98:

1. $194i$. 2. $34, 65, a^2+b^2$. 3. $5\pm 7i$.
4. (b) $\{ac+bd+i(bc-ad)\}/(c^2+d^2)$.
5. $(5-6i)(3+2i)$; 71^2+43^2; 18^2+4^2; 29^2+14^2.
6. $ac-bd+i(bc+ad)$; $ac+bd+i(bc-ad)$.
7. $i, -1, -i, 1, i, -1, -i, 1$. 8. $\cos(\theta+\phi)+i\sin(\theta+\phi)$.
9. (i) $(x+iy\sqrt{7})(x-iy\sqrt{7})$.
 (ii) $(x+\frac{1}{2}y+i\frac{1}{2}y\sqrt{3})(x+\frac{1}{2}y-i\frac{1}{2}y\sqrt{3})$; (iii) $a-\frac{1}{2}b\pm i\frac{1}{2}b\sqrt{3}$ as (ii);
 (iv) $a+4b\pm i2b$; (v) $x-5\pm i\sqrt{5}$; (vi) $\sqrt{7}\pm ix$.
10. $x^2-2ax+a^2-b^2=0$. 11. $x^2+4x+13=0$.
12. $[(ac-bd)e-(ad+bc)f]^2+[(ac-bd)f+(ad+bc)e]^2$ and others.
13. $2, \frac{1}{2}, \frac{1}{2}$.
14. $\frac{1}{4}\cdot\frac{1}{x}-\frac{1}{8}\cdot\frac{1}{x-2i}-\frac{1}{8}\cdot\frac{1}{x+2i}=\frac{1}{4}\cdot\frac{1}{x}-\frac{1}{4}\cdot\frac{x}{x^2+4}$.
15. $\frac{1}{x-2}+\frac{2}{x-1-3i}+\frac{2}{x-1+3i}=\frac{1}{x-2}+\frac{4x-4}{x^2-2x+10}$.

Examples 99:

1. (i) 234, $x^4-14x^3+79x^2-234x+338=0$;
 (ii) 256; $x^5-14x^4+83x^3-256x^2+406x-260=0$.
2. (i) 7; (ii) -2; (iii) 0; $1, -1, i, -i$.
4. 3; 7; -5; (i) -21; (ii) -13. 5. (i) $\frac{7}{5}$; (ii) $\frac{19}{25}$; (iii) 83.
6. $x^2+2ax+a^2-b^2=0$. 7. (i) $5q$; (ii) 0.

Examples 100:

4. (iii) $z'=r(\cos\theta-i\sin\theta)$; $2r^3\cos 3\theta$.
10. (i) circle, centre origin, radius 1;
 (ii) circle, centre (1, 0), radius 2;
 (iii) circle, centre (a, b), radius c.

(iv) line through O at 60° with x-axis;
(v) line through (0, 1) at 45° with x-axis.

11. Two sides of triangle greater than third, or equal if triangle is straight line.

13. (i) $|z| \leqslant 2$; (ii) $|z-1-3i| \geqslant 2$; (iii) $z=k(1+3i)$.

Examples 101:

2. $[a^2-b^2, 2ab]$. 3. $1, -1$. 4. $2a^3-6ab^2$.

6. $\left[\dfrac{a^2-b^2}{a^2+b^2}, \dfrac{2ab}{a^2+b^2}\right]$. 7. $3, -4$. 8. $2, 5$.

9. $[1, 0]$.

10. (i) $[a^2-b^2, 2ab]$; (ii) $[a^2-b^2, -2ab]$; (iii) $[a^2+b^2, 0]$;
(iv) $[1, 0;]$ (v) $\left[\dfrac{a^2-b^2}{a^2+b^2}, \dfrac{-2ab}{a^2+b^2}\right]$.

Examples 102:

1. $2\sin\theta\cos\theta$; $\cos^3\theta-3\cos\theta\sin^2\theta$; $4(\cos^3\theta\sin\theta-\cos\theta\sin^3\theta)$.

2. $\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$; $\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}$; $\cos\dfrac{6\pi}{3}+i\sin\dfrac{6\pi}{3}$,
or $-\frac{1}{2}+\dfrac{i\sqrt{3}}{2}$, $-\frac{1}{2}-\dfrac{i\sqrt{3}}{2}$, 1.

3. $\operatorname{cis}\dfrac{\pi}{3}$, $\operatorname{cis}\pi$, $\operatorname{cis}\dfrac{5\pi}{3}$.

4. $\cos^6\theta=\frac{1}{32}\{\cos 6\theta+6\cos 4\theta+15\cos 2\theta+20\}$.

5. $2\cos 5\theta$, $2i\sin 5\theta$.

8. (i) -1; (ii) $\cos 2\theta+i\sin 2\theta$; (iii) $\cos 2\theta+i\sin 2\theta$;
(iv) $\cos 8\theta+i\sin 8\theta$.

Examples 103:

3. $a+ib=(-a+\frac{1}{3}b\sqrt{3})\omega+(-a-\frac{1}{3}b\sqrt{3})\omega^2$, ω being $\frac{1}{2}(-1+i\sqrt{3})$.

4. If $\omega=\frac{1}{2}(-1+i\sqrt{3})$, $5+7i=5+\dfrac{7\sqrt{3}}{3}+\dfrac{14\sqrt{3}}{3}\omega$,

$$a+ib=a+\frac{b\sqrt{3}}{3}+\frac{2b\sqrt{3}}{3}\omega.$$

10. (i) 0; (ii) $3\omega-3$. 11. $A=B=C=\frac{1}{3}$; $\frac{1}{3}\cdot\dfrac{2x-1}{x^2-x+1}$.

Examples 104:

1. (i) $\frac{1}{2}(e^{2i\theta}+e^{-2i\theta})$; (ii) $\dfrac{1}{2i}(e^{3i\theta}-e^{-3i\theta})$; (iii) e^{ipt}
(iv) $e^{ipt}\left(\dfrac{A}{2}+\dfrac{B}{2i}\right)+e^{-ipt}\left(\dfrac{A}{2}-\dfrac{B}{2i}\right)$.

3. $C=\frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta\{\sin\frac{1}{2}\theta+\sin(n+\frac{1}{2})\theta\}$,
$S=\frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta\{\cos\frac{1}{2}\theta-\cos(n+\frac{1}{2})\theta\}$.

5. $\cosh 2x=\cosh^2 x+\sinh^2 x$; $\sinh 2x=2\sinh x\cosh x$;
$\cosh 3x=4\cosh^3 x-3\cosh x$.

Examples 105:

1. (i) $e^2(\frac{1}{2}+\frac{1}{2}i\sqrt{3})$; (ii) $e^2(\cdot 0706+i\,.\,\cdot 9975)$;
(iii) $-e$;
or (i) $3\cdot 6946+i\,.\,6\cdot 398$; (ii) $\cdot 5217+i\,.\,7\cdot 370$;
(iii) $-2\cdot 7183$.

2. (i) $1\cdot 6094+i\,.\,\cdot 6435$; (ii) $\cdot 6931+i\,.\,1\cdot 0472$;
(iii) $\cdot 3465-i\,.\,\cdot 7854$.

3. (i) $\log z+\log z'=2\log z$; (ii) $\log 10=2\cdot 3026$.

4. $P=e^{r\cos\theta}\cos(r\sin\theta)$, $Q=e^{r\cos\theta}\sin(r\sin\theta)$.

Examples 106:

1. $x^3-3xy^2+i(3x^2y-y^3)$; $r^3\cos 3\theta+ir^3\sin 3\theta$.

2. $x^2-(y+1)^2+i\,.\,2x(y+1)$;
$r^2\cos 2\theta-2r\sin\theta-1+i(r^2\cos 2\theta+2r\cos\theta)$.

3. $ax-by+i(ay+bx)$; if $a+ib=\rho(\cos\phi+i\sin\phi)$,
product $=r\rho\cos(\theta+\phi)+ir\rho\sin(\theta+\phi)$.

4. $\dfrac{x^2-y^2}{(x^2+y^2)^2}-i\dfrac{2xy}{(x^2+y^2)^2}$; $\dfrac{1}{r^2}\cos 2\theta-i\dfrac{1}{r^2}\sin 2\theta$.

5. $(x^2-y^2)\left\{1+\dfrac{1}{(x^2+y^2)^2}\right\}-i\,.\,2xy\left\{1-\dfrac{1}{(x^2+y^2)^2}\right\}$;
$\left(r^2+\dfrac{1}{r^2}\right)\cos 2\theta+i\left(r^2-\dfrac{1}{r^2}\right)\sin 2\theta$.

6. $x\left(1-\dfrac{1}{x^2+y^2}\right)+iy\left(1+\dfrac{1}{x^2+y^2}\right)$; $r\cos\theta\left(1-\dfrac{1}{r^2}\right)+ir\sin\theta\left(1+\dfrac{1}{r^2}\right)$.

7. z'^2. 8. z^3. 9. $\dfrac{1}{z'}$. 10. z^2.

11. Change to $-6x^2y^2$; z^4. 12. Change to $-y^2$ in denom.; z.

Examples 107:

2. Yes. 4. (i) unit circle centre $(-1, 0)$; (ii) the line $u=0$.

Examples 108:

1. $-\frac{1}{2}-\frac{1}{2}i\sqrt{3}$, 1. 3. $x^2+2x+4=0$; $-1-i\sqrt{3}$.

5. (i) 13; $22^\circ\,37'$. (ii) $\sqrt{2}$; -45°. (iii) 1; $-\frac{1}{2}\alpha$. (iv) $\sqrt{10}$; $5^\circ\,49'$.

6. cis 4θ. 7. $-(\sin 3\theta-i\cos 3\theta)$. 8. (i) $\frac{1}{2}\theta$; (ii) $\frac{1}{2}\pi-\frac{1}{2}\theta$.

9. $\sqrt{(\frac{5}{8})}$. 10. $i\,.\,\dfrac{8ab(a^2-b^2)}{(a^2+b^2)^2}$.

12. (i) $(x+1)\left(x^2+2x\cos\dfrac{2\pi}{5}+1\right)\left(x^2+2x\cos\dfrac{4\pi}{5}+1\right)$;
(ii) $(x-1)(x-\omega)(x-\omega^2)(x+1)(x+\omega)(x+\omega^2)$.

16. (i)) $6pq$, $18p^2q^2$.

CHAPTER XII (pp. 286 to 307)

Examples 109:

1. $S_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$; 250. 2. $S_n = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$.

3. Each term of $\Sigma \frac{1}{n^3}$ is less, except the first term; the second.

7. (i) Convergent; (ii) Divergent; (iii) Divergent; (iv) Convergent; (v) Divergent; (vi) Convergent; (vii) Convergent; (viii) Convergent.

Examples 111:

1. $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4$.
2. $1 + \frac{2}{3}x + \frac{5}{9}x^2 + \frac{40}{81}x^3 + \frac{110}{243}x^4$.
3. $1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \frac{3}{256}x^5$;
   (i) $\frac{8}{3x^2}\left[(1-x)^{\frac{3}{2}} - 1 + \frac{3x}{2}\right]$; (ii) $\frac{32}{3}\left[\frac{1}{2\sqrt{2}} - \frac{1}{4}\right]$.
4. $\frac{32}{3x^2}\left[1 + \frac{1}{4}x - (1+x)^{\frac{1}{4}}\right]$.
5. $-\frac{2 \cdot 5 \cdot 8 \ldots (3n-4)}{3^n \cdot n!}$. 6. $\frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{3^n \cdot n!}$; $\sqrt[3]{2}$.

Examples 112:

1. $2e$. 2. $(x^2+x+1)e^x - 1$. 3. $(x^3+3x^2+x+4)e^x - 4$.
4. $x/(1-x) - 2\log(1-x)$.
5. $m(m-1)x^2(1+x)^{m-2} + mx(1+x)^{m-1} + (1+x)^m - 1$.
6. $m(m-1)(m-2)x^3(1+x)^{m-3} + 3m(m-1)x^2(1+x)^{m-2}$
   $+ mx(1+x)^{m-1} + 4(1+x)^m - 4$.
8. $a=6,\ b=7,\ c=1$; $(x^3+6x^2+7x+1)e^x - 1$.
9. $5(e^x - 1) - \log(1-x)$. 11. $2{\cdot}833$.

Examples 114:

1. $\cdot 6931$. 2. (i) e^x; (ii) e^x; (iii) e^{-x}; (iv) e.

4. (i) 1; (ii) $\frac{1}{3}$. 5. (i) $\frac{1}{x} + \frac{1}{2} - \frac{x}{12} + \frac{x^2}{24}$; (ii) $\frac{1}{2}$.

Examples 115:

2. $\frac{(1+2^9a^9)(1-a+ia\sqrt{3})}{1-2a+4a^2}$.
3. Convergent except for θ an odd multiple of $\frac{1}{2}\pi$.
4. -1. 5. $(1 - x\cos\theta + ix\sin\theta)/(1 - 2x\cos\theta + x^2)$.
7. (i) $(1 - 2x\cos\theta + x^2\cos 2\theta)/(1 - 2x\cos\theta + x^2)^2$;
   (ii) $(2x\sin\theta - x^2\sin 2\theta)/(1 - 2x\cos\theta + x^2)^2$.

Examples 116:

1. $x<2\frac{1}{2}$.

Examples 117:

7. $\frac{9}{16}$. **8.** $\sqrt{2}/\sqrt{3}$. **10.** Divergent if $x=1$; covergent if $x=-1$.

11. log 2.

12. (i) Convergent if $|x|<\frac{3}{5}$; (ii) convergent if $|x|<\frac{1}{5}$; $(1-5x)^{-\frac{1}{5}}$.

TEST PAPERS C (pp. 307 to 315)

C.I:

1. (i) $4x^2-13x+1=0$; (ii) $-4\leqslant y\leqslant 1$.

2. $1+\frac{1}{8}x+\frac{63}{128}x^2$. **3.** $\frac{3}{2}$, $1-i$, $1+i$. **4.** $\cdot 105$.

6. $2\text{ cis}\left(\dfrac{2r+1}{3}\right)\pi$, $(r=0, 1, 2)$; 2.

C.II:

1. (*a*) 1365; (i) 100; (ii) 1360. **2.** (ii) $\frac{5}{31}$, $\frac{5}{11}$.

3. $\dfrac{1}{x}-\dfrac{2}{x+1}+\dfrac{1}{x+2}$. **4.** (i) 4, 8.

6. $\frac{1}{2}(1\pm i\sqrt{3})$, $\frac{1}{5}(3\pm 4i)$.

C.III:

1. $13x^2+9x+3$.

2. (i) $\left[z+\dfrac{1}{\sqrt{2}}(1+i)\right]\left[z-\dfrac{1}{\sqrt{2}}(1+i)\right]\left[z+\dfrac{1}{\sqrt{2}}(1-i)\right]\left[z-\dfrac{1}{\sqrt{2}}(1-i)\right]$

2. (ii) $(z^2+1+z\sqrt{2})(z^2+1-z\sqrt{2})$.

3. $19\left[1+\dfrac{2}{3}+\left(\dfrac{2}{3}\right)^2+\ldots\right]$; $\dfrac{1}{(1-r)(1-ar)}$, $|r|<1$ and $|ar|<1$.

5. (*a*) 18; (*b*) $x=4{\cdot}13y^{-2\cdot 5}$; (*c*) -1 or $\cdot 131$.

6. (i) 0; (ii) $\sqrt[3]{2}$; (iii) $e+1$.

C.IV:

2. (ii) $\frac{1}{2}(3\pm\sqrt{5})$.

3. $-x-\frac{5}{2}x^2-\frac{7}{3}x^3$, $[(-1)^{r-1}-2^r]/r$, $-\frac{1}{2}\leqslant x<\frac{1}{2}$.

4. $1+x^4+\frac{5}{8}x^8+\frac{5}{16}x^{12}$. **5.** $\pm\log_e 2$.

C.V:

1. (i) 3, 6, 12; (ii) $x=2y=\pm 4$, $x=-5y=\pm 5$.

3. $1+\dfrac{1}{2}\cdot\dfrac{x}{3}+\dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{x^2}{3^2}+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{x^3}{3^3}$; $-\dfrac{131}{432}$.

6. $2^{\frac{1}{3}}$; 0, $\dfrac{2\pi}{3}$, $\dfrac{4\pi}{3}$; $\sqrt{2}$, $-\dfrac{5\pi}{12}$.

C.VI:

1. 1st term n^2-n+1, last term n^2+n-1; n odd, mid-term n^2; n even, mid-terms n^2-1, n^2+1.

2. Divides line joining z_1 to z_2 externally, ratio 4 : 3.

3. $1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{16}y^3$; $\cos\theta$, $\frac{3}{2}\cos^2\theta - \frac{1}{2}$, $\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$.

6. $\frac{1}{1-x} - \frac{2}{1+2x} + \frac{2x+1}{1+x^2}$; $|x| < \frac{1}{2}$; -30 and 15.

C.VII:

1. (i) 3 ; (ii) $\left(z + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(z + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$.

2. (i) $(7\lambda^2 + 2\lambda - 3)/(\lambda^2 + 1)$. **3.** 6·164.

4. (ii) Circle, diameter's ends dividing z_1 to z_2 internally and externally in ratio 2 : 1.

5. $-1 < x \leqslant 1$, $-1 \leqslant x < 1$; $-1 < x \leqslant 1$.

6. Put $x = \frac{1}{73}$ and add 2 log 6.

C.VIII:

1. $(a+b)(a+\omega b)(a+\omega^2 b)$; $(a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$.

2. $2\frac{11}{48}$.

3. $n(n-1)\ldots(n-r+1)(-3)^r x^{2n-3r}/r!$; 5th, $7 \cdot 3^{12} \cdot 2^{-5}$.

5. (i) $\frac{1}{x-1} - \frac{x}{x^2+1}$; (ii) $-3x^2 - 10x^3 - 28\frac{1}{2}x^4$, $(3 \cdot 2^r - 2 \cdot 3^r)/r$, $-\frac{1}{3} \leqslant x \leqslant \frac{1}{3}$.

6. (i) $2\log_e 2 - 1$; (ii) $\frac{\sqrt{3}}{2}\log_e(2+\sqrt{3})$.

CHAPTER XIII (pp. 316 to 329)

Examples 118:

1. (i) $2 < x < 5$; (ii) $5 < x$; (iii) $x > \frac{75}{19}$; (iv) $x > \frac{30}{7}$.

2. $\frac{5-\sqrt{17}}{2} < x < 2$ and $3 < x < \frac{5+\sqrt{17}}{2}$.

3. (i) None ; (ii) $\frac{7-\sqrt{7}}{2} < x < 3$ and $4 < x < \frac{7+\sqrt{7}}{2}$.

4. $x < 4$ or $x > 4\frac{2}{5}$. **5.** $-2 > x > -3\frac{2}{3}$. **6.** $-4 < x < -\frac{2}{3}$.

Examples 119:

10. $a = 2b = 3c$; impossible, since $2a = 3b = 5c$ and $a = b = c$ cannot both be true.

Examples 120:

2. $12\sqrt{15}$. **3.** $\frac{49}{20}$. **4.** $\frac{25}{2}$.

Examples 121:

2. 324. **4.** 12, 13. **8.** $\frac{1539}{3125}$.

Examples 125:

1. $\frac{2}{3} < x < 4$. **4.** $-1 < x < 2$ and $x > 3$.

CHAPTER XIV (pp. 330 to 341)

Examples 126:

1. (i) 29; (ii) 15; (iii) 0; (iv) -4; (v) 0; (vi) -45; (vii) 36; (viii) 39; (ix) -10.
2. 2.
3. (i) 0 or 38; (ii) 3 or $\frac{11}{6}$.

Examples 127:

1. (i) -7; (ii) 45; (iii) 13; (iv) 6; (v) -1362; (vi) -210.
2. $a+b+c$.
3. (i) 0, 2 or -2; (ii) 3 or $1\frac{5}{6}$; (iii) 2 or -3.
4. (i) 32; (ii) 12.
5. (i) a, b, or $-(a+b)$; (ii) $2a$, b, $-(a+b)$.
9. 0, 1, -1.
11. $x=1$ or a or a root of $ax^2+a(a+1)x+1=0$.

Examples 128:

1. 9, -16, 9.
2. $-3\frac{1}{11}$, -9.
3. $\dfrac{(1-b)(1-c)}{(a-b)(a-c)}$.
4. Determinant $=2abc(a+b+c)^3$.
5. -550.
8. (i) $(2x+2y-1)(x-2y+3)$; (ii) no factors; (iii) $(x+2y+5)(4x+3y+1)$.
9. $x=1$, $z=3$.
10. $\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}=0$; $l\begin{vmatrix} l & h & g \\ m & b & f \\ n & f & c \end{vmatrix}+m\begin{vmatrix} a & l & g \\ h & m & f \\ g & n & c \end{vmatrix}+n\begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix}=0$.
11. $a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1$. An *odd* number of interchanges of the suffices from the order 1, 2, 3 means the sign is minus.
12. $16xyz(x+y+z)$.

CHAPTER XV (pp. 342 to 357)

Examples 129:

6. 41, 43; 59, 61; 71, 73.

Examples 130:

1. (i) $3^2 . 5^2 . 19$; (ii) $2^2 . 3^4 . 7$; (iii) $5^2 . 7^2 . 11$.
2. (i) and (ii) 3^2; (ii) and (iii) 7; (iii) and (i) 5^2.
3. $3^2 . 5^2 . 7^2 . 11 . 19$.
4. (i) 135; (ii) 911.
6. For 2160; 40, 7440, 20, 4. For 8505; 24, 17472, 12, 4.
10. $2^6 . 127$.

Examples 131:

1. 9, 1, 22.
2. 14.
17. 31, 211, 2311.
19. $2^{18} . 3^9 . 5^4 . 7^3 . 11 . 13 . 17 . 19$.
20. $2^{49} . 3^{23} . 5^{12} . 7^8 . 11^4 . 13^4 . 17^3 . 19^2 . 23^2 . 29 . 31 . 37 . 41 . 43 . 47$.

Examples 132:

1. (i) 2 . 9, 3 . 6, 4 . 13, 5 . 7, 8 . 15, 10 . 12, 11 . 14.
(ii) 2 . 10, 3 . 13, 4 . 5, 6 . 16, 7 . 11, 8 . 12, 9 . 17, 14 . 15.

Examples 133:

1. 3. 2. 6. 3. 12.
4. (i) 6; (ii) 6, 13; (iii) none. 5. 29. 6. 20, 112.
7. 68. 8. 38. 9. 27. 10. 49. 11. 96.
12. 47.

Examples 134:

1. 3, 5. 2. 3, 5. 3. 2, 7. 4. 3, 3, roots equal.
5. 1, 5. 6. 3, 4. 7. 3, 5. 8. 9, 60.

Examples 135:

10. (*a*) 129. 14. 41. 15. 29.
16. It is $18(y^3-y)$.
19. $\left(\sum_1^{10} n\right)^2 - \left(\sum_1^{9} n\right)^2$ and $\left(\sum_1^{100} n\right)^2 - \left(\sum_1^{99} n\right)^2$.

TEST PAPERS D (pp. 357 to 366)

D.I:

1. (i) 1, 1·58 (approximately);
(ii) $(1+x+x^2+x^3+x^4)(1-x+x^2-x^3+x^4)$; (iii) 7.
4. (i) 2^n. 6. 1·86.

D.II:

3. $2\left\{\left(\frac{y-1}{y+1}\right)+\frac{1}{3}\left(\frac{y-1}{y+1}\right)^3+\frac{1}{5}\left(\frac{y-1}{y+1}\right)^5+\dots\right\}$.
5. $\frac{1}{2}Q^2$, $-\frac{5}{6}QR$.

D.III:

1. 2, -3, 2. 2. ·100335.
4. (i) $(b^2-2ac)/a^2$; (ii) $(3abc-b^3)/a^3$. 6. ·995.

D.IV:

1. 14, 20, 2, 5 or -6, -20, 2, -5. 2. -2, 2, $\frac{2}{3}$, $\frac{2}{5}$.
5. (ii) 1, $-1\pm 2i$. 6. (i) $3^{\frac{3}{2}}-2$; (ii) $1+\frac{1}{x}+e^x\left(1-\frac{1}{x}\right)$.

D.V:

2. (i) $\frac{5}{34}$. 3. 3·094.
4. (i) $\left(\frac{a-b}{b}\right)-\frac{1}{2}\left(\frac{a-b}{b}\right)^2+\frac{1}{3}\left(\frac{a-b}{b}\right)^3\dots$, $0<a\leqslant 2b$;
(ii) $+\left(\frac{a-b}{a}\right)+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3\dots$, $a<2b$.

5. (i) $-\frac{2182}{243}$; (ii) $1\cdot0039968$.

6. $$\frac{-a^2(a_1-a_2)(a_2-a_3)(a_3-a_1)}{a_1a_2a_3(a-a_1)^2(a-a_2)^2(a-a_3)^2}.$$

D.VI:

1. (a) the second. 2. (i) convergent; (ii) divergent.
3. $1, \frac{1}{6}$. 4. $\alpha^2+2\alpha-14,\ -\alpha^2-3\alpha+14$.
6. $B_1=\frac{1}{6},\ B_2=\frac{1}{30},\ B_3=\frac{1}{42}$.

D.VII:

2. $7<x<8$ and $9<x<10$. 4. $-\frac{5}{x}+\left(\frac{1}{x}-\frac{5}{x^2}\right)\log(1-x)$.
5. $(a-b)(b-c)(c-a)$.

D.VIII:

4. $$\tfrac{1}{2}(4\log 2+\log 3+\log 5+\log 7)+\frac{1}{3361}+\frac{1}{3}\cdot\frac{1}{(3361)^2}+\frac{1}{5}\cdot\frac{1}{(3361)^3}+\dots.$$
5. $\pi=3\cdot139$. 6. $\begin{vmatrix} a_0x+a_1 & a_1x+a_2 \\ a_1x+a_2 & a_2x+a_3\end{vmatrix}=0$.

D.IX:

1. $a>0,\ b^2<4ac$. (i) Range $-\infty$ to ∞ less 0 to $4(p-1)$;
(ii) $0\leqslant y\leqslant 1+p^2$.
3. $1, 3, 1$. 5. (i) $2/\sqrt{3}$; (ii) $e+1$; (iii) $[(1+x)\log(1+x)-x]/x^2$.
6. $A_1=g(a_1)/(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)$, etc.;
begin with
$$\frac{A}{(x-a_1)^2}+\frac{B}{(x-a_1)}+\frac{A_3}{x-a_3}+\dots,$$
and then
$$A=g(a_1)/(a_1-a_3)(a_1-a_4)\dots(a_1-a_n),$$
$$B=\frac{\left\{g'(a_1)-g(a_1)\left[\frac{1}{a_1-a_3}+\frac{1}{a_1-a_4}+\dots\right]\right\}}{(a_1-a_3)(a_1-a_4)\dots(a_1-a_n)};$$
$B_1=1,\ B_2=a_4-a_1,\ B_3=(a_4-a_1)(a_4-a_2)$,
$B_4=(a_4-a_1)(a_4-a_2)(a_4-a_3)$.

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